TOPICS IN AUTOMORPHIC FORMS

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Abstract. Does the space of cusp forms $S_k(N,\chi)$ feature a basis of theta series? We will work through Eichlers solution to this question for $k \geq 2$ and N square free. This will involve studying orders in quaternion algebras, theta series and traces of Hecke operators.

1. Introduction

Historically, the main motivation for studying modular forms came from their connnection to representation numbers of quadratic forms. Let us make this a little more precise. Given a positive definite quadratic form in $2k$ variables and integer coefficients. One is interested in understanding the numbers

$$a(n;Q) = \sharp\{\mathbf{x} \in \mathbb{Z}^{2k} \mid n = Q(\mathbf{x})\}.$$

These can be studied by forming the series[1]

$$\theta(z;Q) = \sum_{n=0}^{\infty} a(n;Q)e(nz).$$

It can be shown, that there is $N = N(Q) \in \mathbb{N}$ and a character $\chi = \chi(Q)$ such that $\theta(\cdot;Q) \in M_k(N,\chi)$ is a modular form of weight k, level N and nebentypus ϵ. The upshot is, that the space $M_k(N,\chi)$ is finite dimensional and one can choose a clever basis in terms of Eisenstein series and cusp forms. Expanding $\theta(\cdot;Q)$ in this basis and comparing coefficients of the Fourier expanisons leads interesting asymptotic and sometimes explicit formulae for the numbers $a(n;Q)$. (See Exercise 1 for an explicit example.) On the level of Dirichlet series this can be seen as decomposing $L(s;Q) = \sum_{n\in\mathbb{N}} a(n;Q)n^{-s}$ into a sum of eulerian Dirichlet series.

It is well known that the graded algebra $\bigoplus_{k\geq 4} M_k(1,\mathrm{Id})$ is generated by the Eisenstein series E_4 and E_6. Here

$$E_k(z) = 1 + \frac{(2\pi)^k}{i^k\zeta(k)\Gamma(k)}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e(nz).$$

Furthermore the dimensions of the spaces $M_k(1,\mathrm{Id})$ as well as $S_k(1,\mathrm{Id})$ are in general well understood. To illustrate this let us just recall the probably most

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[1]We use the common notation $e(x) = e^{2\pi i x}$.

famous example $S_{12}(1,\mathrm{Id}) = \mathbb{C}\Delta$ for

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} = \sum_{n>0} \tau(n)e(nz) = e(z) - 24e(2z) + 252e(3z) + \dots.$$

On the other hand Hecke [4, Satz 46] showed that there is a harmonic polynomial P_8 of degree 8, such that

$$\theta(z; P_8, Q_8) = \Delta(z). \tag{1}$$

Here

$$Q_8 = \frac{1}{2}\sum_{r=1}^{8} x_r^2 + \frac{1}{2}\left(\sum_{r=1}^{8} x_r\right)^2 - x_1x_2 - x_2x_8$$

and $\theta(z; P_8, Q_8)$ is a generalised theta series associated to Q_8 with weights P_8. We will discuss such theta series in more details in Section 3 below.

This still begs the question if any other modular form of given weight, level and nebentypus can be decomposed into theta series and if so, which class of theta series needs to be considered. More precisely one can ask which spaces $S_k(N,\chi)$ have a basis of generalised theta series and how such a basis can be parametrised. If $S_k(N,\chi)$ does not feature a basis in terms of theta series, one can further ask which parts of the space fail to be spanned by theta series and how they can be broken down into pieces that are better understood.

This circle of ideas is sometimes referred to as the basis problem and can be viewed as a predecessor of the Jacquet-Langlands correspondence. Our objective in this course is to reproduce Eichler's solution to the basis problem in case of square free level. Let us conclude this introduction by giving a brief historic overview.

- In [4] Hecke made the following 2 conjectures, which can be viewed as the beginning of the basis saga.
  (1) There is an explicit basis for $S_2(p,\mathrm{Id})$ obtained from theta series. More precisely quaternionic theta series attached to a maximal order in the quaternion algebra with discriminant p.
  (2) The action of the Hecke-operators T_n for $(n,p) = 1$ on the space of theta series is given by a matrix which can be arithmetically defined given the maximal order.
- In [1] Brandt constructs certain matrices, the so called Brandt matrices, associated to maximal orders in quaternion algebras. It will turn out that these are the matrices foreseen by Hecke in the second point above.
- In [2] Eichler proves the (slightly modified) first part of Hecke's conjecture.
- In [3] Eichler used generalised theta series and Brandt matrices associated to so called Eichler orders (these are not necessarily maximal) to solve the basis problem for $S_k(N,\mathrm{Id})$ where N is square-free. (Note that Hecke already used these generalised theta series to represent Δ as described above. Further Hecke tested his conjecture for $S_k(p)$ with $p+1 \mid 24$.)

- For non square-free level or non-trivial nebentypus the story becomes more complicated. A complete solution to the basis problem in full generality has been achieved in [5].

Remark 1.1. The fact that a space of modular forms can be spanned by theta functions is a nice insight but not the core of the matter. It is the second part of Hecke's conjecture that is essential. This is because it provides a way of constructing Hecke eigenfunctions on the level of quadratic forms. Exercise 2 should illustrate this.

Exercise 1. Let $Q(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and set $\sigma_s(n) = \sum_{d|n} d^s$. Show the identity

$$a(n;Q) = 8\left(\sigma_1(n) - \delta_{4|n} 4\sigma_1\left(\frac{n}{4}\right)\right),$$

originally due to Jacobi, and deduce $a(n;Q) \geq 1$ for all $n \geq 1$, which is Lagrange's celebrated four square theorem. Here one can use $\theta(\cdot;Q) \in M_2(4,\mathrm{Id})$ and $M_2(4,\mathrm{Id}) = \langle P - 4P(4\,\cdot), P - 2P(2\,\cdot)\rangle_{\mathbb{C}}$, for

$$P(z) = 1 - 24\sum_{n\geq 1}\sigma_1(n)e(nz).$$

Proof. Set $P_2 = P - 2P(2\,\cdot)$ and $P_4 = P - 4P(4\,\cdot)$. It is easy to compute the following table:

n	$a(n;Q)$	$a_{P_2}(n)$	$a_{P_4}(n)$
0	1	-1	-3
1	8	-24	-24

Next we make the ansatz $\theta(z;Q) = aP_2 + bP_4$, getting the two linear equations

$$a + b = -\frac{1}{3} \text{ and } a + 3b = -1.$$

The solution is $a = 0$, $b = -\frac{1}{3}$, so that

$$\sum_{n=0}^{\infty} a(n;Q)e(nz) = \theta(z;Q) = -\frac{1}{3}P_4 = 1 + 8\sum_{n=1}^{\infty}[\sigma_1(n) - 4\delta_{4|n}\sigma_1(\frac{n}{4})]e(nz).$$

Comparing coefficients completes this exercise. ∎

Remark 1.2. Let Q be the sum of four squares. Then the result in the previous exercise implies

$$\begin{aligned} L(s;Q) &= \sum_{n\in\mathbb{N}} a(n;Q)n^{-s} = 8\sum_{n\in\mathbb{N}}[\sigma_1(n) - 4\delta_{4|n}\sigma_1(\frac{n}{4})]n^{-s} \\ &= 8(1 - 4^{1-s})\zeta(s)\zeta(1-s). \end{aligned}$$

Exercise 2. Show that the space $M_{4l}(1,\mathrm{Id})$ can be spanned by theta series $\theta(\cdot\,,Q_A)$ for positive definite quadratic forms Q_A in $8l$ variables such that $\det(A) = 1$.

Proof. We start by noting that, for two quadratic forms Q_1 and Q_2, we have

$$\theta(z; Q_1)\theta(z; Q_2) = \theta(z; Q_1 \oplus Q_2).$$

The upshot is that it is enough to represent the generators of the algebra $\bigcup_{l>1} M_{4l}(1, \mathrm{Id})$ by theta series. In the case at hand this implies that we need to generate E_4 and Δ.

Let Q_8 be as above. Then

$$0 \neq \theta(z; Q_8) \in M_4(1, \mathrm{Id}) = \mathbb{C}E_4.$$

In particular $E_4 = C_4\theta(z; Q_8)$.[2]

Now note that the first Fourier coefficient of $E_{12}(z)$ is not an integer. Thus, we must have

$$\theta(z; Q_8 \oplus Q_8 \oplus Q_8) = C_{12}E_{12}(z) + C_\Delta \Delta \text{ with } C_\Delta \neq 0.$$

It is now a theorem due to Siegel, that $E_{12}(z) = \theta_{\mathrm{gen}(\mathbb{Q}_8 \oplus Q_8 \oplus Q_8)}(z)$, where $\theta_{\mathrm{gen}(\mathbb{Q}_8 \oplus Q_8 \oplus Q_8)}(z)$ is the genus theta series, which itself is a linear combination of theta series. In particular, we can write

$$\Delta = \frac{\theta(z; Q_8 \oplus Q_8 \oplus Q_8) - C_{12}\theta_{\mathrm{gen}(\mathbb{Q}_8 \oplus Q_8 \oplus Q_8)}(z)}{C_\Delta}$$

and we are done. $\square$

[2]Even more, up to equivalence there is only one positive definite quadratic form with determinant 1 in 8 variables and one can show that $a(n; Q_8) = 240 \cdot \sigma_3(n)$. In 16 variables there is more than one equivalence class of positive definite quadratic forms of determinant 1. However, the space of modular forms of weight 8 and level 1 is still one dimensional. In particular $\theta(z; Q_8 \oplus Q_8) = C_8E_8$, more precisely $a(n; Q_8 \oplus Q_8) = 480 \cdot \sigma_7(n)$.

2. A crash course in modular forms

2.1. Some basic hyperbolic geometry. The group $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathbb{C}\cup\{\infty\}$ via Möbius transformations

$$g\cdot z = \frac{az+b}{cz+d} \text{ for } z\in\mathbb{C} \text{ and } g\cdot\infty = \frac{a}{c}.$$

Here $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$. One checks, that $\mathrm{Im}(g\cdot z) = \frac{\det(g)\,\mathrm{Im}(z)}{|cz+d|^2}$. In particular, we observe that $\mathrm{SL}_2(\mathbb{R})$ and more generally $\mathrm{GL}_2^+(\mathbb{R})$ preserve the upper half plane

$$\mathbb{H} = \{z = x+iy \in \mathbb{C} \mid y > 0\}$$

and its closure $\overline{\mathbb{H}} = \mathbb{H}\cup\mathbb{R}\cup\{\infty\}$. Given $f\colon \mathbb{H}\to\mathbb{C}$ we define

$$[f|_k\alpha](z) = \det(\alpha)^{k-1}(cz+d)^{-k}f(\alpha z).$$

We distinguish four types of matrices in $\mathrm{PSL}_2(\mathbb{R})$. These are

- The identity 1.
- **Parabolic** matrices u. These are matrices with $|\mathrm{tr}(u)| = 2$ and $u \neq 1$. They are distinguished by having exactly one fixed point on $\mathbb{R}\cup\{\infty\}$. Furthermore, they can be conjugated to a matrix acting by $z\mapsto z+c$ for some $c\in\mathbb{R}$.
- **Elliptic** matrices k. These are matrices satisfying $|\mathrm{tr}(k)| < 2$. They have exactly one fixed point in $\mathbb{H}$ and can be conjugated to an element in SO_2.
- **Hyperbolic** matrices a. These are matrices satisfying $|\mathrm{tr}\, a| > 2$. They have exactly two fixed points on $\mathbb{R}\cup\{\infty\}$ and are conjugate to a matrix acting by $z\mapsto cz$ for some $c\in\mathbb{R}_{>0}\setminus\{1\}$.

These types are preserved under conjugation, so that we use the same classification for conjugacy classes.

We say a subgroup $\Gamma\subset\mathrm{SL}_2(\mathbb{R})$ acts discretely on $\mathbb{H}$, if

$$\sharp\{\gamma\in\Gamma \mid \gamma(A)\cap B\neq\emptyset\} < \infty$$

for all compact sets $A, B\subset\mathbb{H}$.

A fundamental domain $\mathcal{F}\subset\mathbb{H}$ for Γ is a set such that $\Gamma\overline{\mathcal{F}} = \mathbb{H}$ and for $z_1, z_2\in\mathcal{F}^\circ$ we have $\Gamma z_1\cap\Gamma z_2 = \emptyset$. The fundamental domains we encounter can and will be chosen such that they are connected, simply connected and measurable.

A fixed point $a\in\mathbb{R}\cup\{\infty\}$ of a parabolic element $\gamma\in\Gamma$. Is called a cusp of Γ. Two cusps are considered to be equivalent if they lie in the same Γ-orbit. Often two equivalent cusps are not further distinguished. (Convention: If we speak about $\alpha\in\mathbb{R}\cup\{\infty\}$ being a cusp of Γ we mean exactly the point α. However when we use $\mathfrak{a}$ to denote a cusp of Γ we are talking about a full equivalence class of cusps.) A scaling matrix $\sigma\in\mathrm{SL}_2(\mathbb{R})$ for a cusp x is a matrix satisfying $\sigma x = \infty$.

Important examples for such subgroups are

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, N \mid c \right\}.$$

These are finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Indeed the index is given by

$$[\mathrm{SL}_2(\mathbb{Z})\colon \Gamma_0(N)] = N \prod_{p \mid N} (1 + \frac{1}{p}).$$

Furthermore a set of inequivalent cusps is given by

$$C(N) = \left\{ \frac{u}{v} \colon v \mid N,\, (u,v) = 1,\, u \mod (v, \frac{N}{v}) \right\}.$$

Given a cusp x and a corresponding scaling matrix $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ there is $h = h(x) \in \mathbb{N}$, called the width of the cusp x, such that

$$\sigma \Gamma_0(N)_x \sigma^{-1} = \left\{ \pm \begin{pmatrix} 1 & hn \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Note that the width h depends only on the equivalence class of x and is independent of the choice of the scaling matrix. One can check that for $x = \frac{u}{v} \in C(N)$ one has $h = \frac{N}{(v^2, N)}$. In particular, if N is square-free we have $h = \frac{N}{v}$.

Further, let $\mathcal{F}_1 = \{z \in \mathbb{H}\colon |z| > 1,\, |\mathrm{Re}(z)| < \frac{1}{2}\}$ be the standard fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. We can choose a fundamental domain $\mathcal{F}_N$ for $\Gamma_0(N)$ by taking a union of $N \prod_{p \mid N} (1 + \frac{1}{p})$ suitably chosen translates of $\mathcal{F}_1$.

We equip $\mathbb{H}$ with the measure $d\mu(z) = \frac{dxdy}{y^2}$, which is $\mathrm{SL}_2(\mathbb{R})$ invariant. Note that

$$\mathrm{Vol}(\mathcal{F}_1, \mu) = \frac{\pi}{3}.$$

So far we have introduced the hyperbolic plane via the upper half plane model. However, sometimes it is useful to work in the Poincare disc. We set $\mathbb{D} = \{w = x + iy \in \mathbb{C}\colon |w| < 1\}$. We have a conformal equivalence given by $\mathbb{H} \to \mathbb{D}$, $z \mapsto \rho.z$, for

$$\rho = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

The hyperbolic volume on $\mathbb{D}$ is given by $d\nu(w) = \frac{4dxdy}{(1+|w|^2)^2}$ and the automorphism group of $\mathbb{D}$ is given by $\mathrm{SU}(1,1)/\{\pm 1\} = \rho\, \mathrm{SL}_2(\mathbb{R})/\{\pm 1\} \rho^{-1}$.

2.2. **Fast track to Modular forms.** A Dirichlet character χ modulo N can be extended to a character of $\Gamma_0(N)$ via $\chi(\gamma) = \chi(d)$, for $\gamma = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N)$.

A function $f\colon \mathbb{H} \to \mathbb{C}$ is called a modular form of weight k, level N and nebentypus χ, if f is holomorphic on $\mathbb{H}^* = \mathbb{H} \cup \{\text{cusps of } \Gamma_0(N)\}$ and satisfies

$$f(\gamma z) = \chi(\gamma)(cz + d)^k f(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

We write $M_k(N,\chi)$ for the set of all modular forms of weight k, level N and nebentypus χ. If f further vanishes at all cusps, then we call f a cusp form. We write $S_k(N,\chi)$ for the set of all cusp forms of weight k, level N and nebentypus χ.

Since $\Gamma_0(N)$ contains the subgroup

$$N(\mathbb{Z}) = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z} \right\},$$

we have $f(z+1) = f(z)$. Therefore, we can expand $f \in M_k(N,\chi)$ in a Fourier expansion at ∞:

$$f(z) = \sum_{n \geq 0} a_f(n) e(nz).$$

Note that, if $f \in S_k(N,\chi)$ then $a_f(0) = 0$. Similarly, f has Fourier expansions at any other cusp of $\Gamma_0(N)$ and $f \in S_k(N,\chi)$ if and only if the $0th$ Fourier coefficient vanishes at every cusp.

There is an important family of commuting operators on the space $S_k(N,\chi)$. These are the Hecke operators, which we define in an ad-hoc manner by

$$[T_n f](z) = \frac{1}{n} \sum_{\substack{ad=n, \\ (a,N)=1}} \chi(a) a^k \sum_{b \mod d} f\left(\frac{az+b}{d}\right).$$

As mentioned above these operators commute and satisfy the interesting relation

$$T_m T_n = \sum_{\substack{d|(m,n), \\ (d,N)=1}} \chi(d) d^{k-1} T_{\frac{nm}{d^2}}.$$

Furthermore, for $(n,N) = 1$, these operators are essentially self adjoint (with respect to the Petersson inner product). More precisely,

$$\langle T_n f, g \rangle = \chi(n) \langle f, T_n g \rangle.$$

Thus we can find a basis of $S_k(N)$ consisting of joint eigenfunctions of all Hecke-operators T_n with $(n,N) = 1$. Such a basis will be called Hecke eigenbasis and its elements are referred to as Hecke eigenforms. We deonte the n-th Hecke eigenvalue of a Hecke eigenform f by $\lambda_f(n)$.

We define the unramified Hecke-algebra $\mathbf{T}(N) = \mathbf{T}_\chi(N)$ to be the $\mathbb{Z}$ algebra generated by $\{T_p \colon (p,N) = 1\}$. For $\chi = 1$ we can view this as a subring of operators acting on (holomorphic) functions $\Gamma_0(N)\backslash\mathbb{H} \to \mathbb{C}$. We can similarly consider the $\mathbb{Q}$-algebra $\mathbf{T}(N) \otimes_{\mathbb{Z}} \mathbb{Q}$. So far we have seen that $\mathbf{T}(N)$ is a commutative semisimple algebra and $S_k(N,\chi)$ is a finite dimensional $\mathbf{T}(N)$-module.

We now wish to enlarge the algebra $\mathbf{T}(N)$ by the ramified Hecke operators T_p for $p \mid N$. To get a satisfying theory for this larger algebra we need to take care of some technical issues. Define the space of oldforms by

$$S_k^\flat(N,\chi) = \langle \{ z \mapsto f(dz) \colon f \in S_k(N',\chi) \text{ for } Q \mid N' \mid N, N' \neq N \text{ and } d \mid \frac{N}{N'} \} \rangle,$$

where Q is the conductor of χ. In particular we can view χ as a Dirichet character modulo N' as long as $Q \mid N'$. Further we define $S_k^\sharp(N,\chi) = (S_k^\flat(N,\chi))^\perp$. We call elements of the latter space newforms. It is easy to check, that the Hecke-operators respect the just defined spaces. Thus, we can choose a Hecke eigenbasis for the space of newforms. Elements of this basis will be called Hecke newforms in all what follows. These Hecke newforms have some remarkable properties which more than justifies the construction made above.

Each Hecke newform satisfies $a_f(1) \neq 0$, so that we usually normalise them by requiring $a_f(1) = 1$. Further, if two Hecke newforms $f, g \in S_k^\sharp(N,\chi)$ satisfy $\lambda_f(n) = \lambda_g(n)$ for all (but finitely many) $(n,N) = 1$, then $f = g$. Thus, in particular a Hecke newform f is automatically an eigenfunction of all the Hecke operators. Including those with $(n,N) \neq 1$. Even more, we have

$$f(z) = \sum_{n\in\mathbb{N}} \lambda_f(n) e(nz).$$

Example 2.1. Let us have a look at the space $S_k(37, \mathrm{Id})$. Note that, since $S_2(1,\mathrm{Id}) = \{0\}$ and 37 is a prime number we have $S_k(37,\mathrm{Id}) = S_k^\sharp(37,\mathrm{Id})$. With the help of the LMFDB we can have a closer look. We find that $\dim_{\mathbb{C}} S_k(37,\mathrm{Id}) = 2$. The normalised newforms are given by

$$f_{37.2.a.a}(z) = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 + \dots,$$
$$f_{37.2.a.b}(z) = q + q^3 - 2q^4 - q^7 - 2q^9 + 3q^{11} + \dots,$$

where we use the standard notation $q = e(z)$. Furthermore there is exactly one (up to normalisation) Eisenstein series of level 37, weight 2 and trivial nebentypus. Thus one has $\dim_{\mathbb{C}} M_2(37,\mathrm{Id}) = 3$.

Exercise 3. Compute the Fourier coefficients (at ∞) of the (suitably normalised) Eisenstein series in $M_2(37,\mathrm{Id})$.

Proof. We start by observing that the function

$$P(z) = 1 - 24 \sum_{n\in\mathbb{N}} \sigma_1(n) e(nz) \tag{2}$$

is well defined for $z \in \mathbb{H}$ and the sum converges nicely. This resembles the Fourier-expansion of the weight k Eisenstein series. However in the weight 2 case the standard argument that produces modularity does not work since the sum

$$\frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z},\\ (c,d)=1}} (cz+d)^{-2}$$

is not absolutely convergent. By arguing carefully one can still show that

$$[P|_2\gamma](z) = P(z) - \frac{6i}{\pi} \cdot \frac{c}{cz+d}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Thus it does not define a modular form for the full modular group. However, a simple computation shows that

$$P_{37}(z) = P(z) - 37 \cdot P(37z) \in M_2(37, \mathrm{Id}).$$

It is known that $\dim_{\mathbb{C}} M_2(37, \mathrm{Id}) = 3$ and the example above shows two linearly independent cusp forms. Thus, P_{37} must be the missing element. It is an Eisenstein series because it obviously has non-vanishing constant term at ∞. The Fourier coefficients are given by

$$P_{37}(z) = -36 - 24 \sum_{n \in \mathbb{N}} \left[\sigma_1(n) - 37 \delta_{37|n} \sigma_1\left(\frac{n}{37}\right) \right] e(nz).$$

$\square$

3. Theta series reloaded

The goal of this section is to proof the modularity of certain generalised theta series. The content is fairly standard. We mainly follow the exposition in [6] and skip over several details.

Let

$$\mathcal{SP}_{2k} = \{A \in \mathrm{Mat}_{2k\times 2k}(\mathbb{Z}) \mid A > 0, \text{ symmetric} \\ \text{and the diagonal entries of } A \text{ are even}\}.$$

Given $A \in \mathcal{SP}_{2k}$ we can write $A = S^t S$ for a real matrix S. Note that one always has $(-1)^k \det(A) \equiv 0, 1 \mod 4$. Furthermore, there is a minimal $N_A \in \mathbb{N}$ such that $N_A A^{-1} \in \mathcal{SP}_{2k}$. This integer is called the level of A. To $A \in \mathcal{SP}_{2k}$ we can associate the quadratic form

$$Q_A(\mathbf{x}) = \frac{1}{2}\mathbf{x}^t A \mathbf{x}.$$

Let $P_l(\mathbf{x}) = P(x_1, \dots, x_{2k})$ be a homogeneous polynomial of degree l. We call P_l harmonic, if

$$\Delta P_l = 0.$$

To $z \in \mathbb{H}$ and a solution $\mathbf{r}$ of the congruence

$$A\mathbf{r} \equiv 0 \mod N_A$$

we associate the (generalised) theta series

$$\theta_{P_l,Q_A}(z,\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}^{2k}} P_l\left(S(n+N^{-1}\mathbf{r})\right) e\left(\frac{1}{2}(n+N^{-1}\mathbf{r})^t A(n+N^{-1}\mathbf{r})z\right).$$

We are now going to study the transformation behaviour under the action of $\Gamma_0(1)$ on z. Note, that this group is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Lemma 3.1.

$$\left(\theta_{P_l,Q_A}(\cdot,\mathbf{r})|[T]_{k+l}\right)(z) = e\left(\frac{1}{2N^2}\mathbf{r}^t A\mathbf{r}\right)\theta_{P_l,Q_A}(z,\mathbf{r}).$$

Proof. This follows directly form the definition of θ_{P_l,Q_A} and the observation

$$\left(\theta_{P_l,Q_A}(\cdot,\mathbf{r})|[T]_{k+l}\right)(z) = \theta_{P_l,Q_A}(z+1,\mathbf{r}).$$

□

We now prove a kind of functional equation which is crucial for our further analysis.

Lemma 3.2.

$$\sum_{\mathbf{m}\in\mathbb{Z}^{2k}} P_l(S(\mathbf{m}+\mathbf{x}))e\left(\frac{1}{2}(\mathbf{m}+\mathbf{x})^t A(\mathbf{m}+\mathbf{x})z\right)$$
$$= \frac{i^k z^{-k-l}}{\sqrt{|\det(A)|}} \sum_{\mathbf{m}\in\mathbb{Z}^{2k}} P_l(S^{-t}\mathbf{m})e\left(\frac{-\mathbf{m}^t A^{-1}\mathbf{m}}{2z} + \mathbf{m}^t\mathbf{x}\right)$$

For $l = 0$ the proof is a straight forward application of Poisson summation and the direct evaluation of the Gaussian integral. In most modern expositions one deduces the general case by applying suitable integral operators. However, we will follow the exposition of [3] here.

Proof. We set

$$f(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^{2k}} P_l(S(\mathbf{m}+\mathbf{x}))e\left(\frac{1}{2}(\mathbf{m}+\mathbf{x})^t A(\mathbf{m}+\mathbf{x})z\right).$$

Since the sum converges absolute and uniformly (for fixed $z \in \mathbb{H}$) f determines a continuous function with period 1 in each argument. Thus we obtain the Fourier expansion

$$f(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^{2k}} c_{\mathbf{m}} e(\mathbf{m}^t\mathbf{x}) \tag{3}$$

with coefficients

$$\begin{aligned} c_{\mathbf{m}} &= \int_0^1 \dots \int_0^1 f(\mathbf{x})e(-\mathbf{m}^t\mathbf{x})dx_1 \dots dx_{2k} \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^{2k}} \int_0^1 \dots \int_0^1 P_l(S(\mathbf{m}+\mathbf{x}))e\left(\frac{1}{2}(\mathbf{m}+\mathbf{x})^t A(\mathbf{m}+\mathbf{x})z - \mathbf{m}^t\mathbf{x}\right) dx_1 \dots dx_{2k} \\ &= \int_{\mathbb{R}} \dots \int_{\mathbb{R}} P_l(S\mathbf{x})e\left(\frac{1}{2}\mathbf{x}^t A\mathbf{x}z - \mathbf{m}^t\mathbf{x}\right) dx_1 \dots dx_{2k}. \end{aligned}$$

Since the integrand is holomorphic we can make the change of variables $\mathbf{x} - z^{-1}A^{-1}\mathbf{m} \mapsto \mathbf{x}$ and shift the contour back to the real line. Thus we obtain

$$c_{\mathbf{m}} = e(-\frac{1}{2}\mathbf{m}^t A^{-1}\mathbf{m}z^{-1}) \int_{\mathbb{R}} \dots \int_{\mathbb{R}} P_l(S\mathbf{x} + z^{-1}S^{-t}\mathbf{m})e\left(\frac{1}{2}\mathbf{x}^t A\mathbf{x}z\right) dx_1 \dots dx_{2k}.$$

By holomorphicity in z it is enough to compute the integral for $z = iy$ with $y > 0$. We make the change of variables $\sqrt{y}S\mathbf{x} \mapsto \mathbf{x}$, which yields

$$c_{\mathbf{m}} = \frac{e(-\frac{1}{2}\mathbf{m}^t A^{-1}\mathbf{m}z^{-1})}{y^{k+l}\det(S)} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} P_l(\sqrt{y}\mathbf{x} - iS^{-t}\mathbf{m})e\left(\frac{i}{2}\mathbf{x}^t\mathbf{x}\right) dx_1 \dots dx_{2k}.$$

Using the spectral theory of the Laplacian on the unit sphere we can expand

$$P_l(\sqrt{y}\mathbf{x} - iS^{-t}\mathbf{m}) = (-i)^l P_l(S^{-t}\mathbf{m}) + \sum_{\deg(Q)>0} b_Q Q(\mathbf{x})$$

in harmonic polynomials Q, which are orthogonal with respect to integration over the unit sphere. We now switch to polar coordinates. Note that by orthogonality we have

$$\int_{S^{2k}} Q(\mathbf{x})d\mathbf{x} = 0$$

for all Q with $\deg(Q) > 0$. We arrive at

$$c_{\mathbf{m}} = \mathrm{Vol}(S^{2k})\frac{i^k z^{-k-l}}{\det(S)}e(-\frac{1}{2}\mathbf{m}^t A^{-1}\mathbf{m}z^{-1})P_l(S^{-t}\mathbf{m})\int_0^\infty e^{-\pi r^2} r^{2k-1}dr.$$

Inserting this formula for the Fourier coefficients in (3) completes the proof. ☐

Corollary 3.3. *We have*

$$\theta_{P_l,Q_A}(-z^{-1},\mathbf{r}) = \frac{i^k(-z)^{k+l}}{\sqrt{|\det(A)|}}\sum_{A\mathbf{l}\equiv 0 \ mod \ N_A} \psi(\mathbf{r},\mathbf{l})\theta_{P_l,Q_A}(z,\mathbf{l}),$$

for $\psi(\mathbf{r},\mathbf{l}) = e\left(\frac{\mathbf{l}^t A\mathbf{r}}{N_A^2}\right)$.

Proof. There is a one to one correspondence between $\mathbf{m} \in \mathbb{Z}^{2k}$ and $\mathbf{n} \in \mathbb{Z}^{2k}$ such that $A\mathbf{n} \equiv 0 \bmod N_A$, which is explicitely given by $\mathbf{n} = N_A A^{-1}\mathbf{m}$. We find

$$\theta_{P_l,Q_A}(-z^{-1},\mathbf{r}) = \frac{i^k(-z)^{k+l}}{\sqrt{|\det A|}}\sum_{A\mathbf{n}\equiv 0 \bmod N_A} P_l(S\mathbf{n}N_A^{-1})e\left(\frac{\mathbf{n}^t A\mathbf{n}}{2N_A^2}z + \frac{\mathbf{n}^t A\mathbf{r}}{N_A^2}\right).$$

The result follows directly from rearranging te $\mathbf{n}$-sum. ☐

Remark 3.4. The functions $\mathbf{l} \mapsto \psi(\mathbf{l},\mathbf{r})$ are actually characters of the finite abelian group

$$\{\mathbf{l} \bmod N_A \colon A\mathbf{l} \equiv 0 \bmod N_A\}.$$

This group has order $\det(A)$, so that we have the important identity

$$\sum_{\substack{\mathbf{l} \bmod N_A,\\ A\mathbf{l}\equiv 0 \bmod N_A}} \psi(\mathbf{1},\mathbf{r}) = \delta_{\mathbf{r}\equiv 0}\det(A).$$

With this sort of functional equation at hand the modularity proof succeeds as usual. For completeness we recall the details.

Let d be odd and $(c,d) = 1$. Then we define

$$G(c,d) = \sum_{x \mod d} e\left(c\frac{x^t Ax}{d}\right).$$

We will need the following lemma, which we recall without providing a proof.

Lemma 3.5. *Let* $\left(\frac{c}{d}\right)$ *be the Jacobi-Symbol and put*

$$\epsilon_d = \left(\frac{-1}{d}\right)^{\frac{1}{2}} = \begin{cases} 1 & \text{if } d \equiv 1 \mod 4,\\ i & \text{if } d \equiv 3 \mod 4.\end{cases}$$

Then, if $A \in \mathcal{SP}_n$, we have

$$G(c,d) = \left(\frac{\det(A)}{d}\right)\left[\epsilon_d\left(\frac{c}{d}\right)\sqrt{d}\right]^n.$$

Lemma 3.6. *Let $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $d \equiv 1$ mod 2, $d > 0$ and $c \equiv 0$* mod N_A. *Then*

$$\theta_{P_l,Q_A}(\tau z;\mathbf{r}) = \left(\frac{(-1)^k\det(A)}{d}\right) e\left(\frac{ab\mathbf{r}^tA\mathbf{r}}{2N_A^2}\right)(cz+d)^{k+l}\theta_{P_l,Q_A}(z;a\mathbf{r}).$$

Proof. We set $\gamma = \tau w$ and see

$$d\cdot\gamma z = b - \frac{1}{dz-c}.$$

For γ we can compute

$$\theta_{P_l,Q_A}(\gamma z;\mathbf{r}) = \sum_{\substack{\mathbf{g} \bmod dN_A,\\ \mathbf{g}\equiv\mathbf{r} \bmod N_A}} e\left(\frac{b\mathbf{g}^tA\mathbf{g}}{2dN_A^2}\right)\underbrace{\sum_{\mathbf{n}\equiv\mathbf{g} \bmod dN_A} P_l(N^{-1}S\mathbf{n})e\left(\frac{d\mathbf{n}^tA\mathbf{n}}{2d^2N_A^2}\cdot\frac{-1}{dz-c}\right)}_{=d^l\theta_{P_l,Q_{dA}}(-\frac{1}{dz-c};\mathbf{g})}$$

$$= \frac{i^k(-dz+c)^{k+l}d^l}{\sqrt{|\det(dA)|}}\sum_{\substack{\mathbf{g} \bmod dN_A,\\ \mathbf{g}\equiv\mathbf{r} \bmod N_A}} e\left(\frac{b\mathbf{g}^tA\mathbf{g}}{2dN_A^2}\right)\sum_{dA\mathbf{l}\equiv 0 \bmod dN_A}\psi(\mathbf{g},\mathbf{l})\theta_{P_l,Q_{dA}}(dz-c,\mathbf{l})$$

$$= \frac{i^k(-dz+c)^{k+l}d^l}{\sqrt{|\det(dA)|}}\sum_{dA\mathbf{l}\equiv 0 \bmod dN_A}\theta_{P_l,Q_{dA}}(dz,\mathbf{l})\sum_{\substack{\mathbf{g} \bmod dN_A,\\ \mathbf{g}\equiv\mathbf{r} \bmod N_A}} e\left(\frac{b\mathbf{g}^tA\mathbf{g}+2\mathbf{g}^tA\mathbf{l}-c\mathbf{l}^tA\mathbf{l}}{2dN_A^2}\right).$$

Here we applied the previously proven functional equation as well as the observation that $c\mathbf{n}^tA\mathbf{n} \equiv c\mathbf{l}^tA\mathbf{l}$ mod $2dN_A^2$ as $c \equiv 0$ mod N_A. Some elementary manipulations show that

$$\sum_{\substack{\mathbf{g} \bmod dN_A,\\ \mathbf{g}\equiv\mathbf{r} \bmod N_A}} e\left(\frac{b\mathbf{g}^tA\mathbf{g}+2\mathbf{g}^tA\mathbf{l}-c\mathbf{l}^tA\mathbf{l}}{2dN_A^2}\right) = e\left(\frac{a\mathbf{r}^tA\mathbf{l}}{N_A^2}\right)\sum_{\substack{\mathbf{g} \bmod dN_A,\\ \mathbf{g}\equiv\mathbf{r} \bmod N_A}} e\left(\frac{b\mathbf{g}^tA\mathbf{g}}{2dN_A^2}\right). \quad (4)$$

In particular, this sum only depends on $\mathbf{l}$ mod N_A. Thus, we obtain

$$\theta_{P_l,Q_A}(\tau z;\mathbf{r}) = \frac{i^k(dz^{-1}+c)^{k+l}}{d^k\sqrt{|\det(A)|}}\sum_{A\mathbf{l}\equiv 0 \bmod N_A}\theta_{P_l,Q_A}(\frac{-1}{z},\mathbf{l})$$

$$\cdot\sum_{\substack{\mathbf{g} \bmod dN_A,\\ \mathbf{g}\equiv\mathbf{r} \bmod N_A}} e\left(\frac{b\mathbf{g}^tA\mathbf{g}+2\mathbf{g}^tA\mathbf{l}-c\mathbf{l}^tA\mathbf{l}}{2dN_A^2}\right).$$

Applying the functional equation again we find

$$\theta_{P_l,Q_A}(\tau z;\mathbf{r}) = \frac{(-1)^l(cz+d)^{k+l}}{d^k|\det(A)|}\sum_{A\mathbf{h}\equiv 0 \bmod N_A}\theta_{P_l,Q_A}(z,\mathbf{h})$$
$$\cdot\sum_{A\mathbf{l}\equiv 0 \bmod N_A}\sum_{\substack{\mathbf{g} \bmod dN_A,\\ \mathbf{g}\equiv\mathbf{r} \bmod N_A}} e\left(\frac{b\mathbf{g}^tA\mathbf{g}+2\mathbf{g}^tA\mathbf{l}-2\mathbf{l}^tA\mathbf{h}+c\mathbf{l}^tA\mathbf{l}}{2dN_A^2}\right).$$

The $\mathbf{l}$-sum can be evaluated using (4) and contributes $|\det(A)|\delta_{\mathbf{h}\equiv -a\mathbf{r} \bmod N_A}$. Thus we have

$$\theta_{P_l,Q_A}(\tau z;\mathbf{r}) = \frac{(-1)^l(cz+d)^{k+l}}{d^k}\theta_{P_l,Q_A}(z,a\mathbf{r})\cdot\sum_{\substack{\mathbf{g} \bmod dN_A,\\ \mathbf{g}\equiv\mathbf{r} \bmod N_A}} e\left(\frac{b\mathbf{g}^tA\mathbf{g}}{2dN_A^2}\right).$$

The evaluation of the $\mathbf{g}$-sum can be reduced to Lemma 3.5. ☐

Given $c\equiv 0,1 \bmod 4$ we define a completely multiplicative character χ_c as follows. For positive, odd d we set $\chi_c(d)=\left(\frac{c}{d}\right)$. Further put $\chi_c(-1)=\mathrm{sgn}(c)$ and

$$\chi_c(2)=\begin{cases}1 & \text{if } c\equiv 1 \bmod 8,\\ -1 & \text{if } c\equiv 5 \bmod 8,\\ 0 & \text{if } (c,2)>1.\end{cases}$$

Proposition 3.7. *We have $\theta_{P_l,Q_A}(\cdot\,,0)\in M_{k+l}(N_A,\chi_{(-1)^k\det(A)})$. If $l>0$, then $\theta_{P_l,Q_A}(\cdot\,,0)$ is cuspidal.*

Proof. It is an easy exercise to check the required transformation behaviour of $\theta_{P_l,Q_A}(\cdot\,,0)$ is easily deduced from Lemma 3.6. In order to check holomorphicity and vanishing at the cusps it is enough to consider the functions $\theta_{P_l,Q_A}(\cdot\,,\mathbf{r})$ at ∞. We leave the details to the reader. ☐

Exercise 4. Find the harmonic polynomial P_8 such that (1) holds.

Proof. For now take P to be some harmonic polynomial. If $\deg(P)=0$, in other words $P=c$, then

$$\theta(z;P;Q_8)=cE_4(z) \tag{5}$$

On the other hand if $\deg(P)>0$, then $\theta(z;P;Q_8)$ is cuspidal. In particular, if $0<\deg(P)<8$, then $\theta(z;P;Q_8)=0$ since there are no cusp forms of the corresponding weight and level. Furthermore, if $\deg(P)=8$ then $\theta(z;P;Q_8)\in\mathbb{C}\Delta$. Thus it suffices to find P for which $\theta(z;P;Q_8)\neq 0$.

Next let us note the following triviality

$$\theta(z;P,Q_8)=\sum_{n\geq 0}B_P(n)e(nz) \text{ for } B_p(n)=\sum_{\substack{\mathbf{x}\in\mathbb{Z}^8,\\ Q_8(\mathbf{x})=n}}P(\mathbf{x}).$$

This of course implies

$$\theta(z; P+\tilde{P}, Q_8) = \theta(z; P; Q_8) + \theta(z; \tilde{P}; Q_8)$$

On the other hand, our discussion above implies

$$B_p(n) = \begin{cases} 240 \cdot c \cdot \sigma_3(n) & \text{if } P = c, \\ 0 & \text{if } 1 \leq \deg(P) < 8. \end{cases}$$

We make the ansatz

$$P_8(\mathbf{x}) = \tilde{P}_8(u)[Q_8(\alpha)\cdot Q_8(\mathbf{x})]^4 \text{ for } u = \frac{\mathbf{x}^t A_{Q_8}\alpha}{2\sqrt{Q_8(\mathbf{x})Q_8(\alpha)}}$$

and some $\alpha \in \mathbb{Z}^8$ to be specified soon. Here $\tilde{P}_8$ is a certain even Polynomial of degree 8. We can write

$$\tilde{P}_8(u) = u^8 + \sum_{\rho=1}^{7} c_\rho H_\rho(u) - w_8,$$

for Legendre-like-Polynomials H_ρ of degree ρ.[3] Since P_8 is orthogonal to the constant function one determines $w_8 = 2^{-7}$.

Using our remarks above we compute

$$\begin{aligned} B_{P_8}(n) &= (Q_8(\alpha)\cdot n)^4 \sum_{Q_8(\mathbf{x})=n} (u^8 - w_8) \\ &= 2^{-8} \sum_{Q_8(\mathbf{x})=n} [\mathbf{x}^t A_{Q_8}\alpha]^8 - 2^{-7}\cdot 240 \cdot \sigma_3(n)(Q_8(\alpha)\cdot n)^8. \end{aligned}$$

To see that our generalised theta function does not vanish we only need to look at the first Fourier coefficient. We now choose α such that $Q_8(\alpha) = 1$. Thus we get

$$2^8 B_8(1) = \underbrace{\sum_{Q_8(\mathbf{x})=1} \left[\mathbf{x}^t A_{Q_8}\alpha\right]^8}_{\geq 2\cdot(2Q_8(\alpha))^8} - 2^5\cdot 15 \geq 2^9 - 2^5\cdot 15 = 2^5(16-15) = 2^5 > 0.$$

In the first step we used that the $\mathbf{x}$ sum includes $\mathbf{x} = \alpha, -\alpha$ and we drop all the rest by positivity. Thus we have seen that $B_8(1) \geq \frac{1}{8}$ which implies non-vanishing.[4] □

[3] If we were working in 3 variables these would really be just Legendre polynomials. However, in our case they are determined (up to constant) by their property of being polynomials of degree ρ and by the differential equation

$$(1-u^2)H_\rho'' - 7uH_\rho' + \rho(6+\rho)H_\rho = 0.$$

[4] Working more precisely one can get $B_8(1) = \frac{9}{16}$ on the nose.

4. The Eichler trace formula

In order to establish a ready to use Eichler trace formula of quite some generality we follow the master's thesis of Fabian Völz, who in turn follows [8]. For background on the Bergman kernel and complex analysis in multiple variables we refer to the book [7]. Throughout this section $k \geq 2$ is a fixed even integer. At some point we will further assume $k > 2$ to avoid complications.

4.1. **A quick tour through reproducing kernel Hilbert spaces.** Let X be a set and $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space of complex valued functions on X.

Definition 4.1. A function $K\colon X \times X \to \mathbb{C}$ is called a **reproducing kernel** of H if $K(\cdot, x) \in H$ for all $x \in X$ and $f(x) = \langle f, K(\cdot, x) \rangle$ for all $x \in X$ and all $f \in H$. If H admits a reproducing kernel it is called a **reproducing kernel Hilbert space**.

Example 4.1. A toy example of a reproducing kernel Hilbert space is as follows. Let $X = \{1, \ldots, n\}$ and $H = \{f\colon X \to \mathbb{C}\} = \mathbb{C}^n$ with the standard inner product

$$\langle f, g \rangle = \frac{1}{n} \int_X f(x)\overline{g(x)} dx = \frac{1}{n} \sum_{i=1}^{n} f(i)\overline{g(i)}.$$

A reproducing kernel for H is given by

$$K(i, j) = n\delta_j(i) \text{ with } \delta_j(i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Indeed $K(\cdot, j) = n\delta_j \in H$ and $\langle f, K(\cdot, j) \rangle = n\langle f, \delta_j \rangle = f(j)$.

Lemma 4.2. *Let H be a reproducing kernel Hilbert space. Then the reproducing kernel K is unique and satisfies*

$$K(x, y) = \overline{K(y, x)}.$$

Furthermore $K(x, x) = 0$ if and only if $f(x) = 0$ for all $f \in H$.

Proof. The first claim follows since

$$K(x, y) = \langle K(\cdot, y), K(\cdot, x) \rangle = \overline{\langle K(\cdot, x), K(\cdot, y) \rangle} = \overline{K(y, x)}.$$

The second claim follows from the inequality

$$K(x, x) = \langle K(\cdot, x), K(\cdot, x) \rangle \geq \frac{|\langle f, K(\cdot, x) \rangle|^2}{\|f\|^2} = \frac{|f(x)|^2}{\|f\|^2} > 0,$$

which holds for every $f \in H$. □

Proposition 4.3. *H is a reproducing kernel Hilbert space if and only if $E_x\colon H \to \mathbb{C}, f \mapsto f(x)$ is continuous for all $x \in X$.*

Proof. Suppose H is a reproducing kernel Hilbert space. Then

$$|E_x(f)| = |\langle f, K(\cdot, x)\rangle| \leq \|K(\cdot, x)\| \|f\| = \sqrt{K(x,x)}\|f\|.$$

On the other hand, if E_x is continuous, by the Riesz representation theorem there is $g_x \in H$ such that $E_x = \langle \cdot, g_x\rangle$. The reproducing kernel is now obviously given by $K(y, x) = g_x(y)$. □

Proposition 4.4. *Let H be a reproducing kernel Hilbert space. If $\mathcal{B}_H$ is an orthonormal basis of H, then*

$$K(x, y) = \sum_{\phi \in \mathcal{B}_H} \phi(x)\overline{\phi(y)}$$

is the reproducing kernel of H. If $H \subset J$, where J is a larger Hilbert space. Then

$$\pi_K\colon J \to H, (\pi_K f)(x) = \langle f, K(\cdot, x)\rangle$$

is a well defined projection operator.

Proof. We first expand

$$K(\cdot, y) = \sum_{\phi \in \mathcal{B}_H} \langle K(\cdot, y), \phi\rangle \phi, \tag{6}$$

where the convergence is understood with respect to the norm of H. Recall that the evaluation maps E_x are continuous, so that

$$K(x, y) = \sum_{\phi \in \mathcal{B}_H} \overline{\langle \phi, K(\cdot, y)\rangle}\phi(x) = \sum_{\phi \in \mathcal{B}_H} \overline{\phi(y)}\phi(x).$$

The fact that π_K is well defined is obvious. Further, from (6) we see that the image of π_K is indeed H. However, by the reproducing property we have

$$[\pi_K^2 f](x)\langle\langle f, K(\cdot, *)\rangle, K(*, x)\rangle = \langle f, K(\cdot, x)\rangle = [\pi_K f](x).$$

Thus π_K is an idempotent and we are done. □

We now continue our discussion on reproducing kernels but restrict ourselves to a very specific case connected to modular forms. For integrable $f, g\colon \mathbb{H} \to \mathbb{C}$ we define

$$\langle f, g\rangle_k = \int_{\mathbb{H}} f(z)\overline{g(z)} \operatorname{Im}(z)^k d\mu(z).$$

Further we define the Hilbert space

$$L_k^2(\mathbb{H}) = \{f\colon \mathbb{H} \to \mathbb{C}\colon \langle f, f\rangle_k < \infty\}/\sim.$$

This space contains the subspace of $H_k^2(\mathbb{H})$ of holomorphic functions.

Proposition 4.5. *The space $H_k^2(\mathbb{H})$ is a reproducing kernel Hilbert space.*

Proof. We start by proving the following inequality:

$$\sup_{z\in\overline{B_\epsilon(z_0)}}|f(z)| \leq C_{z_0}\|f\cdot \mathbb{1}_{\overline{B_{3\epsilon}(z_0)}}\|_{2,k}. \tag{7}$$

Indeed this follows from the Taylor expansion $f(z)=\sum_{n=0}^\infty b_n(z-z_0)^n$ around z_0 as follows. Observe that

$$\int_{B_\epsilon(z_1)} f(z)\operatorname{Im}(z)^2 d\mu(z) = \int_0^{2\pi}\int_0^\epsilon \sum_{n=0}^\infty b_n r^{n+1}e^{in\theta}drd\theta = \pi\epsilon^2 b_0 = \pi\epsilon^2 f(z_1).$$

With this at hand we are almost done. Indeed

$$\begin{aligned}
|f(z_1)| &\leq \frac{1}{\pi\epsilon^2}\int_{B_{3\epsilon}(z_0)}|f(z)\operatorname{Im}(z)^2|d\mu(z)\\
&\leq \frac{\sup_{z\in B_{3\epsilon}(z_0)}|\operatorname{Im}(z)^{\frac{k}{2}-2}|}{\pi\epsilon^2}\int_{B_{3\epsilon}(z_0)}|f(z)\operatorname{Im}(z)^{\frac{k}{2}}|d\mu(z)\\
&\leq \frac{\sup_{z\in B_{3\epsilon}(z_0)}|\operatorname{Im}(z)^{\frac{k}{2}-2}|}{\pi\epsilon^2}\left(\int_{B_{3\epsilon}(z_0)}1d\mu(z)\right)^{\frac{1}{2}}\|f\cdot\mathbb{1}_{\overline{B_{3\epsilon}(z_0)}}\|_{2,k}.
\end{aligned}$$

This implies the continuity of the evaluations E_x. Thus, if $H_k^2(\mathbb{H})$ is a Hilbert space, then it is automatically reproducing. We are left with showing completeness of $H_k^2(\mathbb{H})$. To do so we take a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$. Since $L_k^2(\mathbb{H})$ is complete, this converges to some $f\in L_k^2(\mathbb{H})$. However, due to our inequality above, this convergence is uniform on compacta, so that f is holomorphic. (This follows from Morera's theorem and some analytic contiuation argument.) □

Remark 4.6. In a similar way we can introduce the spaces $H_k^p\subset L_k^p(\mathbb{H})$. It turns out these are Banach spaces for $1\leq p\leq\infty$.

We denote the reproducing kernel of $H_k^2(\mathbb{H})$ by K_k. Our next goal is to compute K_k explicitly. We do so in several steps.

Lemma 4.7. *For $f\in H_k^2(\mathbb{H})$ and $\alpha\in\mathrm{GL}_2(\mathbb{R})^+$ we have*

$$\|f|_k\alpha\|_{2,k} = \det(\alpha)^{\frac{k}{2}-1}\|f\|_{2,k}.$$

In particular, $f|_k\alpha\in H_k^2(\mathbb{H})$.

Proof. This follows from the simple computation

$$\begin{aligned}
\|f|_k\alpha\|_{2,k}^2 &= \int_{\mathbb{H}}|[f|_k\alpha](z)|^2\operatorname{Im}(z)^k d\mu(z) = \int_{\mathbb{H}}|f(\alpha z)|\frac{\det(\alpha)^{2k-2}\operatorname{Im}(z)^k}{|cz+d|^{2k}}d\mu(z)\\
&= \det(\alpha)^{k-2}\int_{\mathbb{H}}|f(\alpha z)|\operatorname{Im}(\alpha z)^k d\mu(z) = \det(\alpha)^{k-2}\|f\|_{2,k}^2.
\end{aligned}$$

□

Lemma 4.8. *For $\alpha \in \mathrm{GL}_2(\mathbb{R})^+$ we have*

$$K_k(\alpha z, \alpha w) = \det(\alpha)^{-k} j(\alpha, z)^k \overline{j(\alpha, w)}^k K_k(z, w).$$

Proof. We define

$$K_k^{(\alpha)}(z, w) = \det(\alpha)^k j(\alpha, z)^{-k} \overline{j(\alpha, w)}^{-k} K_k(\alpha z, \alpha w).$$

The idea is to check, that $K_k^{(\alpha)}$ is a reproducing kernel for $H_k^2(\mathbb{H})$, since then the claimed equality follows by uniqueness.

Thus we compute:

$$\begin{aligned}\langle f, K_k^{(\alpha)}(\cdot, w)\rangle &= \det(\alpha)^{k-1} j(\alpha, w)^{-k} \langle f|_k \alpha^{-1}, K_k(\cdot, \alpha w)\rangle \\ &= \det(\alpha)^{k-1} j(\alpha, w)^{-k} [f|_k \alpha^{-1}](\alpha w) = f(w).\end{aligned}$$

Here we simply used the definition of the inner product (as integral) and the reproducing property of K_k (justified by the previous lemma). Observing that, due to the previous lemma, $K_k^{(\alpha)}(\cdot, w)$ is an element of $H_k^2(\mathbb{H})$ finishes the proof. $\square$

Proposition 4.9. *There is a constant C_k such that*

$$K_k(z, w) = C_k \left(\frac{z - \overline{w}}{2i}\right)^{-k}.$$

Proof. We define

$$\Omega = \{(z, w) \in \mathbb{C}^2 \colon z \in \mathbb{H}, \overline{z - w} \in \mathbb{H}\} = \{(z, w) \in \mathbb{C}^2 \colon \mathrm{Im}(w) > \mathrm{Im}(y)\}$$

and set

$$h \colon \Omega \to \mathbb{C},\ (z, w) \mapsto K_k(z, \overline{z - w}).$$

Note that h is holomorphic in w, since we can write it as $h(z, w) = \overline{K_k(\overline{z} - \overline{w}, z)}$ and K_k is holomorphic in the first argument. We now argue that h is holomorphic in z. To do so we look at $H(z_1, z_2) = K_k(z_1, \overline{z_2 - w})$. It is clear that H is holomorphic in each variable (thus weakly holomorphic as a function $H \colon \mathbb{C}^2 \to \mathbb{C}$). Thus (by Hartog's theorem) $H \colon \mathbb{C}^2 \to \mathbb{C}$ is holomorphic (in any definition one likes). By composing H with $z \mapsto (z, z)$ we see that $h(\cdot, w)$ is holomorphic for each admissible w. To summarise, we have seen that h is holomorphic in both arguments.

Our next goal is to show that $h(z, w) = h(z', w)$ for all $(z, w), (z', w) \in \Omega$. To do so we consider

$$\Phi(\tau) = h(z + \tau, w) - h(z, w),$$

for fixed (z, w) and τ with $\mathrm{Im}(\tau) < \mathrm{Im}(w) - \mathrm{Im}(z)$. Note that by the previous lemma applied with $\alpha = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ we have $\Phi(b) = 0$ for $b \in \mathbb{R}$. Furthermore, Φ is holomorphic in a neighbourhood of the real line. We conclude that Φ is the constant zero function (where it is defined), so that $h(\cdot, w)$ is also constant.

Thus we can define

$$l(w) = h(z_w, w) = K_k(z_w, \overline{z_w - w})$$

for any $z_w \in \mathbb{H}$ such that $(z_w, w) \in \Omega$. Note that l is holomorphic and $l(z - \overline{w}) = K(z, w)$. Using the previous lemma with $\alpha = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ we find

$$l(2aw) = l(aw - \overline{a(-\overline{w})}) = K(aw, a(-\overline{w})) = a^{-k} K_k(w, -\overline{w}) = a^{-k} l(2w),$$

for $a \in \mathbb{R}_+$. Thus, by taking $w = \frac{i}{2}$ we find

$$l(iy) = y^{-k} l(i).$$

Thus, l agrees on the imaginary axis, and thus everywhere, with the function $\frac{z}{i}$. This concludes the proof with the constant $C_k = 2^{-k} l(i)$. $\square$

Theorem 4.10. *We have*

$$K_k(z, w) = \frac{k-1}{4\pi} \left(\frac{z - \overline{w}}{2i} \right)^{-k},$$

for $k \geq 2$.

Proof. We do so by explicitly computing $\langle f_0, K_k(\cdot, w) \rangle$, for $f_0(z) = (z + i)^{-k}$. However, we first have to show that $f_0 \in H_k^2(\mathbb{H})$ for $k \geq 2$. We compute

$$\begin{aligned}
\|f_0\|_{2,k} &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} |x + i(y+1)|^{-2k} y^{k-2} dy dx \\
&= \int_{\mathbb{R}} \int_1^\infty (x^2 + y^2)^{-k} (y-1)^{k-2} dy dx \\
&\leq \int_{\mathbb{R}} \int_1^\infty (x^2 + y^2)^{-k} y^{k-2} dy dx \\
&\leq \int_0^\pi \int_1^\infty r^{1-k} (r \sin(\theta))^{k-2} dr d\theta \\
&= \int_0^\pi \sin(\theta)^{k-2} d\theta \int_1^\infty r^{-k-1} dr < \infty.
\end{aligned}$$

Now we have

$$(2i)^{-k} = f_0(i) = \langle f_0, K_k(\cdot, i) \rangle_k = C_k (2i)^k \int_{\mathbb{H}} (z+i)^{-k} (i - \overline{z})^{-k} \operatorname{Im}(z)^k d\mu(z)$$

One computes that $(z+i)(i-\overline{z}) = -(x^2+(1+y)^2)$ and finds

$$\begin{aligned} C_k^{-1} &= 4^k \int_{\mathbb{R}} \int_{\mathbb{R}_+} (x^2+(y+1)^2)^{-k} y^{k-2} dy dx \\ &= 4^k \int_{\mathbb{R}_+} \frac{y^{k-2}}{(y+1)^{2k}} \int_{\mathbb{R}} \left(\left(\frac{x}{y+1} \right)^2 + 1 \right)^{-k} dx dy \\ &= 4^k \int_{\mathbb{R}_+} \frac{y^{k-2}}{(y+1)^{2k-1}} dy \int_{\mathbb{R}} (s^2+1)^{-k} ds. \end{aligned}$$

Looking up the Beta-function reveals

$$\int_{\mathbb{R}_+} \frac{y^{k-2}}{(y+1)^{2k-1}} dy = B(k-1,k) = \frac{\Gamma(k-1)\Gamma(k)}{\Gamma(2k-1)}.$$

Furthermore, one can see (in many different ways) that

$$\int_{\mathbb{R}} (s^2+1)^{-k} ds = 4\pi \frac{\Gamma(2k-1)}{4^k \Gamma(k)^2} \tag{8}$$

Combining these evaluations we get

$$C_k^{-1} = 4\pi(k-1)^{-1}$$

and we are done. □

Lemma 4.11. *Let $k>2$, then $K_k(\cdot, w) \in H_k^p(\mathbb{H})$ for each $p \in [1,\infty]$.*

Proof. With our explicit formula for $K_k(\cdot,w)$ the L^1-norm can be computed to be

$$\| K_k(\cdot, w) \|_{1,k} = \frac{2^k(k-1)}{4\pi} \operatorname{Im}(w)^{-\frac{k}{2}} B(\frac{k}{2}-1, \frac{k}{2}) \int_0^\pi \sin(\theta)^{k-2} d\theta.$$

This is a nice exercise in integration. (Note that $B(0,x) = \Gamma(0) = \infty$, which makes the assumption $k>2$ necessary.)

Furthermore, $\| K_k(\cdot,w)\|_{\infty,k} = \frac{k-1}{4\pi} \operatorname{Im}(w)^{-\frac{k}{2}}$. The rest follows by interpolation. □

Now, for $k>2$, we can use the Hölder inequality to make sense of the operator

$$\pi_k \colon L_k^p(\mathbb{H}) \to L_k^p(\mathbb{H}),\ f \mapsto [w \mapsto \langle f, K_k(\cdot, w)\rangle].$$

With a little more work one obtains the following theorem which we state without a proof.

Theorem 4.12. *For $k>2$ and any $p \in [1,\infty]$ the operator $\pi_k \colon L_k^p(\mathbb{H}) \to H_k^p(\mathbb{H})$ is a well defined projection.*

After having established the necessary theory for globally symmetric space $\mathbb{H}$ we have to move on to situations closer to the real world. We adjust our definitions slightly. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a discrete subgroup. We define

$$L_k^p(\Gamma\backslash\mathbb{H}) = \{ f \colon \mathbb{H} \to \mathbb{C} \colon f|_k\gamma = f\ \forall \gamma \in \Gamma \text{ and } \|f\|_{p,k} < \infty \},$$

for

$$\|f\|_{p,k}^p = \int_{\Gamma\backslash\mathbb{H}} |f(z)\operatorname{Im}(z)^{\frac{k}{2}}|^p d\mu(z)$$

with the usual modification for $p = \infty$. Note that this is well defined by the transformation behaviour of f and we can replace $\Gamma\backslash\mathbb{H}$ by any fundamental domain $\mathcal{F}$. Further note that for $p = 2$ the norm is defined by the Petersson inner product. As before we let $H_k^p(\Gamma\backslash\mathbb{H}) \subset L_k^p(\Gamma\backslash\mathbb{H})$ be the subspace of holomorphic functions.

Lemma 4.13. *The spaces $H_k^p(\Gamma\backslash\mathbb{H})$ are Banach spaces for $p \in [1,\infty]$ and $H_k^s(\Gamma\backslash\mathbb{H})$ is a reproducing kernel Hilbert space.*

The proof is very similar to the one showing that $H_k^2(\mathbb{H})$ is a reproducing kernel Hilbert space. The only technical difference being that one has to argue using a suitably chosen fundamental domain. We leave the details as an exercise.

Theorem 4.14. *We have $S_k(N) = H_k^\infty(\Gamma_0(N)\backslash\mathbb{H}) = H_k^2(\Gamma_0(N)\backslash\mathbb{H})$.*

Proof. The inclusions $S_k(\Gamma_0(N)\backslash\mathbb{H}) \subset H_k^\infty(\Gamma_0(N)\backslash\mathbb{H}) \subset H_k^2(\Gamma_0(N)\backslash\mathbb{H})$ are obvious since $\Gamma_0(N)$ has finite co-volume and it is an easy exercise to show that $y^{\frac{k}{2}}f(x+iy)$ is bounded on $\mathbb{H}$ for $f \in S_k(N)$. Thus it remains to show the inclusion $H_k^2(\Gamma_0(N)\backslash\mathbb{H}) \subset S_k(N)$. This is done using the Laurent expansion at infinity, which exists for each $f \in H_k^2(\Gamma_0(N)\backslash\mathbb{H})$. Suppose $f(z) = \sum_{-\infty}^{\infty} a_n e(nz)$. Then we compute, for l large enough

$$\begin{aligned}
\infty \geq \|f\|_{2,k} &\geq \iint_{\substack{0\leq \operatorname{Re}(z)\leq 1,\\ l\leq \operatorname{Im}(z)<\infty}} |f(z)\operatorname{Im}(z)^{\frac{k}{2}}|^2 d\mu(z) \\
&= \int_l^\infty \int_0^1 \sum_{m,n\in\mathbb{Z}} a_m\overline{a_n}e^{-2\pi iy(m-n)}e((m-n)x)y^{k-2}dxdy \\
&\geq |a_n|^2 \int_l^\infty e^{-4\pi yn}y^{k-2}dy.
\end{aligned}$$

However, if $k \geq 2$, then the final integral diverges for all $n \leq 0$. Thus the Laurent expansion at ∞ reads $f(z) = \sum_{\mathrm{n}\in\mathbb{N}} a_n e(nz)$. The same exercise can be repeated for all the other cusps and we are done. □

Finally we will construct the reproducing kernel $K_{k,N}$ for $S_k(N)$ explicitly. From now on we take $k > 2$ once and for all. Indeed we set

$$K_{k,N}(z,w) = \frac{1}{2}\sum_{\gamma\in\Gamma_0(N)} [K_k(\cdot,w)|_k\gamma](z).$$

Lemma 4.15. *For $k > 2$ the series defining $K_{k,N}$ is uniformly convergent on any compact subset of $\mathbb{H}\times\mathbb{H}$.*

Note that for $k = 2$ one has to be careful. Similar issues occur for the Eisenstein series E_2.

Proof. Let $g_w \in \mathrm{SL}_2(\mathbb{R})$ be such that $w = g_w i$. Then we have

$$|K_k(z,w)| \operatorname{Im}(z)^{\frac{k}{2}} = |K_k(g_w^{-1}z, i)| \operatorname{Im}(g_w^{-1}z)^{\frac{k}{2}} \operatorname{Im}(w)^{-\frac{k}{2}}.$$

With this at hand we can continue by a standard computation involving the unfolding trick.

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{F}_N} \sum_{\gamma\in\Gamma_0(N)} |K_k(\gamma z, w)| \operatorname{Im}(\gamma z)^{\frac{k}{2}} d\mu(z) \\
&= \frac{1}{2} \sum_{\gamma\in\Gamma_0(N)} \int_{\gamma\mathcal{F}_N} |K_k(z,w)| \operatorname{Im}(z)^{\frac{k}{2}} d\mu(z) \\
&= \int_{\mathbb{H}} |K_k(z,w)| \operatorname{Im}(z)^{\frac{k}{2}} d\mu(z) = \|K_k(\cdot, i)\|_{1,k} \operatorname{Im}(w)^{-\frac{k}{2}}.
\end{aligned}$$

Interchanging sums is justified by the monotone convergence theorem. We conclude that

$$\|K_{k,N}(\cdot,w)\|_{1,k} \leq \|K_k(\cdot,i)\|_{1,k} \operatorname{Im}(w)^{-\frac{k}{2}}.$$

In particular we have seen absolute convergence for fixed w. One can further show that this convergence is locally uniform, so that $K_{k,N}(\cdot,w)$ is holomorph for fixed w. We skip the details here.

For fixed $z_0 \in \mathbb{H}$ we observe, compare to (7), that

$$\sup_{z\in B_\epsilon(z_0)} |K_{k,N}(z,w)| \leq C_{z_0} \|K_{k,N}(\cdot,w)\|_{1,k}.$$

We combine this with our estimate above to establish

$$\sup_{\substack{z\in B_\epsilon(z_0),\\ w\in B_\delta(w_0)}} |K_{k,N}(z,w)| \leq C_{z_0} \sup_{w\in B_\delta(w_0)} [\operatorname{Im}(w)^{-\frac{k}{2}}] \|K_k(\cdot,i)\|_{1,k},$$

for $\epsilon, \delta > 0$ small enough and $z_0, w_0 \in \mathbb{H}$ fixed. Thus the sum is locally uniformly convergent and the statement follows from a standard covering argument. $\square$

Theorem 4.16. *For $k > 2$ the reproducing kernel of $S_k(N)$ is given by $K_{k,N}$.*

Proof. The first step of the proof is to show $K_{k,N}(\cdot,w) \in S_k(N)$ for all $w \in \mathbb{H}$. To do so it is enough to show $K_{k,N}(\cdot,w) \in L_k^\infty(\mathbb{H})$. this is because we already know that $K_{k,N}(\cdot,w) \in H_k^1(\Gamma_0(N)\backslash\mathbb{H})$. By somehow enumerating $\Gamma_0(N)$ we can express $K_{k,N}(\cdot,w) = \lim_{n\to\infty} f_n$ for partial sums f_n of length n. Since $K_k(\cdot,w) \in L_k^\infty(\mathbb{H})$ we have $f_n \in L_k^\infty(\mathbb{H})$.

We now identify $L_k^\infty(\mathbb{H}) = (L_k^1(\mathbb{H}))'$ as usual. Now $(f_n)_{n\in\mathbb{N}}$ corresponds to a sequence of functionals $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n(g) = \langle f_n, g\rangle_k.$$

Our next goal is to show that the sequence x_n is weak-$\star$-convergent. To do so we fix $g \in L^1_k(\mathbb{H})$ and compute

$$\begin{aligned}
x_n(g) = \langle f_n, g\rangle_k &= \int_{\mathbb{H}} \left(\frac{1}{2} \sum_{i=1}^{n} [K_k(\cdot, w)|_k \gamma_i](z) \right) \overline{g(z)} \operatorname{Im}(z)^k d\mu(z) \\
&= \frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{H}} [K_k(\cdot, w)|_k \gamma_i](z) \overline{g(z)} \operatorname{Im}(z)^k d\mu(z) \\
&= \frac{1}{2} \sum_{i=1}^{n} \overline{j(\gamma_i^{-1}, w)}^{-k} \langle K_k(\cdot, \gamma_i^{-1} w), g\rangle_k.
\end{aligned}$$

We now recall the statement of Theorem 4.12, so that we can write

$$\lim_{n\to\infty} x_n(g) = \overline{\frac{1}{2} \sum_{\gamma\in\Gamma_0(N)} [(\pi_k g)|_k \gamma](w)}.$$

Note that the right hand side is nicely behaved for all w. This follows from a thorough treatment of Poincare-treatment series which we omit. Now the Banach-Steinhaus-Theorem tells us that there is $x \in (L^1_k(\mathbb{H}))'$ with $\lim_{n\to\infty} x_n(g) = x$ for all $g \in L^1_k(\mathbb{H})$. Using our identification backwards we find $f \in L^\infty_k(\mathbb{H})$ with $x(g) = \langle f, g\rangle_k$ for all $g \in L^1_k(\mathbb{H})$.

If we show that $f = K_{k,N}(\cdot, w)$ almost everywhere in $\mathbb{H}$, then $K_{k,N}(\cdot, w) \in L^\infty_k(\mathbb{H})$ and the first step is complete. Let us assume the contrary. Then there is a compact set $K \subset \mathbb{H}$ such that

$$N = \{z \in K \colon f(z) \neq K_{k,N}(z, w)\}$$

satisfies $0 < \mu(N) \leq \mu(K) < \infty$. Recall that the partial sums f_n are continuous, thus measurable, and converge pointwise to $K_{k,N}(\cdot, w)$. According to Egorov's theorem there is a closed set A such that $\mu(N \setminus A) \geq \mu(N)/2$ and the convergence of f_n is uniform on A. We conclude that for $g \in L^1_k(\mathbb{H})$ with support in A we have

$$\langle f, g\rangle_k = \langle K_{k,N}(\cdot, w), g\rangle_k.$$

We define

$$G(z) = \operatorname{sgn}(f(z) - K_{k,N}(z, w)) \mathbb{1}_A(z).$$

With this choice we find

$$\int_A |f(z) - K_{k,N}(z, w)| \operatorname{Im}(z)^k d\mu(z) = \langle f - K_{k,N}(\cdot, w), G\rangle_k = 0.$$

In particular $f = K_{k,N}(\cdot, w)$ almost everywhere in A. This is a contradiction to $\mu(A) \geq \mu(N)/2 > 0$ and $A \subset N$. This completes the first step.

For $f \in S_k(N)$ we compute, skipping some of the by now familiar details,

$$\begin{aligned}\langle f, K_{k,N}(\cdot, w)\rangle &= \int_{\mathcal{F}_N} f(z) K_{k,N}(w,z) \operatorname{Im}(z)^k d\mu(z) \\ &= \frac{1}{2}\int_{\mathcal{F}_N} \sum_{\gamma\in\Gamma_0(N)} f(z) K_k(\gamma w, z) j(\gamma, w)^{-k} \operatorname{Im}(z)^k d\mu(z) \\ &= \int_{\mathbb{H}} f(z)\overline{K_k(z,w)} \operatorname{Im}(z)^k d\mu(z) \\ &= \langle f, K_k(\cdot, w)\rangle = (\pi_k f)(w).\end{aligned}$$

Since $f \in S_k(N)$ we have $\pi_k(f) = f$, so that we have shown

$$\langle f, K_{k,N}(\cdot, w)\rangle = f(w).$$

This completes the proof up to the minor point that we still have to justify the exchange of limit and integral above. This follows easily from

$$\begin{aligned}&\sum_{\gamma\in\Gamma_0(N)} \int_{\mathcal{F}_N} |f(z) K_k(\gamma w, z) j(\gamma, w)^{-k} \operatorname{Im}(z)^k| d\mu(z) \\ &\leq \sup_{z_1\in\mathbb{H}} |f(z_1) \operatorname{Im}(z_1)^{\frac{k}{2}}| \sum_{\gamma\in\Gamma_0(N)} \int_{\mathcal{F}_N} |K_k(\gamma^{-1} z_2, w) \operatorname{Im}(\gamma^{-1} z_2)^{\frac{k}{2}}| d\mu(z_2) \\ &= \frac{1}{2}\|f\|_\infty \|K_k(\cdot, w)\|_1 < \infty.\end{aligned}$$

□

4.2. **A rough expansion of traces of Hecke-operators.** Let us now consider the Hecke-operator $T_n \colon S_k(N) \to S_k(N)$. Our ultimate goal is to find a useful expression ot its trace. Recall that since $S_k(N)$ is a finite dimensional inner product space we can fix an orthonormal basis $(f_j)_{j=1,\dots,m}$. We can further assume that the functions f_j are joint eigenfunctions of all Hecke-operators. Note that this requires newform theory if we want to include $(n, N) > 1$. Now we can expand the trace as follows

$$\operatorname{Tr}(T_n) = \sum_{j=1}^m \lambda_{f_j}(n) = \sum_{j=1}^m \langle T_n f_j, f_j\rangle.$$

From this we deduce the following result.

Proposition 4.17. *Let $n \in \mathbb{N}$ and $k \geq 3$. Then*

$$\operatorname{Tr}(T_n) = \frac{n^{k-1}}{2} \int_{\Gamma_0(N)\backslash\mathbb{H}} \sum_{\alpha\in\Delta_{n,N}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k d\mu(z),$$

for

$$\Delta_{n,N} = \{\gamma \in \operatorname{Mat}_2(\mathbb{Z}) \colon N \mid c,\ (a, N) = 1,\ \det(\gamma) = n\}.$$

Proof. Write

$$\Delta_{n,N} = \bigcup_{\substack{ad=n,\\ (a,N)=1,\\ b \mod d}} \Gamma_0(N)\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} = \bigcup_{i=1}^{d} \Gamma_0(N)g_i.$$

Thus

$$[T_n f_j](z) = \sum_{i=1}^{d} \det(g_i)^{k-1} j(g_j,z)^{-k} f(g_i z).$$

We can expand

$$\begin{aligned}
\mathrm{Tr}(T_n) &= \sum_{j=1}^{m} \int_{\mathcal{F}_N} \left(\sum_{i=1}^{d} \det(g_i)^{k-1} j(g_j,z)^{-k} f(g_i z)\right) \overline{f_j(z)} \,\mathrm{Im}(z)^k d\mu(z) \\
&= n^{k-1} \int_{\mathcal{F}_N} \sum_{i=1}^{d} \left(\sum_{j=1}^{m} f(g_i z)\overline{f_j(z)} \right) j(g_j,z)^{-k}\,\mathrm{Im}(z)^k d\mu(z) \\
&= n^{k-1} \int_{\mathcal{F}_N} \sum_{i=1}^{d} K_{k,N}(g_j z, z) j(g_j,z)^{-k}\,\mathrm{Im}(z)^k d\mu(z).
\end{aligned}$$

The statement follows by using the geometric definition of $K_k(N)$. □

This is not yet a useful expression and needs to be refined. Note that with respect to $\gamma^{-1}\alpha\gamma = \alpha$ the integrand satisfies

$$\begin{aligned}
&K_k(\gamma^{-1}\alpha\gamma z, z) j(\gamma^{-1}\alpha\gamma, z)^{-k}\,\mathrm{Im}(z)^k \\
&= K(\alpha\gamma z, \gamma z) j(\gamma, \gamma^{-1}\alpha\gamma z)^{-k} \overline{j(\gamma,z)}^{-k} j(\gamma^{-1}\alpha\gamma, z)^{-k}\,\mathrm{Im}(z)^k \\
&= K(\alpha\gamma z, \gamma z) j(\alpha, \gamma z)^{-k}\,\mathrm{Im}(\gamma z)^k \left(\frac{j(\alpha,\gamma z) j(\gamma, z)}{j(\gamma, \gamma^{-1}\alpha\gamma z) j(\gamma^{-1}\alpha\gamma, z)} \right)^k \\
&= K(\alpha\gamma z, \gamma z) j(\alpha, \gamma z)^{-k}\,\mathrm{Im}(\gamma z)^k.
\end{aligned}$$

Therefore it seems natural to organise the sum y conjugacy classes and arrange the integrals accordingly. However, we first have to overcome some convergence issues that arise when interchanging summation and integration. This is the content of the following section.

Exercise 5. Find an element $\alpha \in \Delta_{n,N}$ such that the obvious orbital integral

$$\int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z, z) j(\alpha, z)^{-k}\,\mathrm{Im}(z)^k d\mu(z).$$

diverges.

4.3. **Interchanging summation and integration.** Let $g_1, \dots, g_l \in \mathrm{SL}_2(\mathbb{Z})$ be such that

$$\mathcal{F}_N = \bigcup_{j=1}^{l} g_j \mathcal{F}_1.$$

Further given a cusp $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ we define $\Delta_{\frac{p}{q}} = \{\gamma \in \Delta_{n,N} \colon \gamma\frac{p}{q} = \frac{p}{q}\}$. We fix some neighbourhood $U_\infty = \{z \in \mathbb{H} \colon \operatorname{Im}(z) > \delta\}$ of ∞ for some $\delta > 1$. This neighbourhood can be transformed into a neighbourhood of an arbitrary cusp via $U_{\frac{p}{q}} = \sigma_{\frac{p}{q}} U_\infty$, for a suitable integer matrix $\sigma_{\frac{p}{q}}\infty = \frac{p}{q}$. Set

$$F_{\frac{p}{q}} = \mathcal{F}_N \cap U_{\frac{p}{q}} \text{ and } F^\circ = \mathcal{F}_N \setminus \bigcup_{\frac{p}{q}} F_{\frac{p}{q}}.$$

The following lemma lays the groundwork for exchanging the sum and integral in the neighbourhood of a cusp.

Lemma 4.18. *Let $\frac{p}{q}$ be a cusp of $\Gamma_0(N)$, then*

$$\int_{F_{\frac{p}{q}}} \sum_{\alpha \in \Delta_{n,N} \setminus \Delta_{\frac{p}{q}}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k d\mu(z)$$
$$= \sum_{\alpha \in \Delta_{n,N} \setminus \Delta_{\frac{p}{q}}} \int_{F_{\frac{p}{q}}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k d\mu(z).$$

Proof. We will proof this for $\frac{p}{q} = \infty$ using the following two facts:

$$\sum_{\substack{\alpha \in \Gamma_\infty \setminus (\Delta_{n,N} \setminus \Delta_\infty)/\Gamma_\infty, \\ \alpha = \begin{pmatrix} * & * \\ c & * \end{pmatrix}}} |c|^{-k} < \infty, \text{ for } k > 2; \tag{9}$$

$$\sum_{n \in Z} ([a+n]^2 + b)^{-l} \ll_l |b|^{-2l}(1 + |b|), \text{ for } l > \frac{1}{2}. \tag{10}$$

The case $\frac{p}{q} \neq \infty$ follows by translating the integrand and adjusting the fundamental domain.

We set

$$S(z) = \sum_{\alpha \in \Delta_{n,N} \setminus \Delta_\infty} |K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k|.$$

We need to show that $S(z)$ converges for all $z \in F_\infty$ and that S is integrable on F_∞.

We write $\Delta_{n,N} = \bigsqcup_{\alpha\in\mathcal{A}} \Gamma_\infty \alpha$ and $\Delta_{n,N}\setminus\Delta_\infty = \bigsqcup_{\alpha\in\mathcal{A}_0}\Gamma_\infty\alpha$. We can arrange these sets such that $\mathcal{A}_0 = \mathcal{A}\setminus\Delta_\infty$. We observe that

$$S(z) = 2\operatorname{Im}(z)^k \sum_{\alpha\in\mathcal{A}_0} |j(\alpha,z)|^{-k} \sum_{m\in\mathbb{Z}} |K_k(\alpha z+m,z)|.$$

By inserting the explicit expression for K_k we find that the inner sum is

$$\sum_{m\in\mathbb{Z}} |K_k(\alpha z+m,z)| = \frac{k-1}{4\pi}2^k \sum_{m\in\mathbb{Z}} |\alpha z+m-\overline{z}|^{-k} \ll_k |\operatorname{Im}(\alpha z-\overline{z})|^{-k}(1+|\operatorname{Im}(\alpha z-\overline{z})|),$$

where we expanded the absolute value and used (10). We use the trivial estimate $\operatorname{Im}(\alpha z-\overline{z}) = \operatorname{Im}(\alpha z)+\operatorname{Im}(z) \geq \operatorname{Im}(z)$ to find

$$S(z) \ll_k (1+\operatorname{Im}(z)) \sum_{\alpha\in\mathcal{A}_0} |j(\alpha,z)|^{-k}$$

We now find a subset $\mathcal{A}_0' \subset \mathcal{A}_0$ such that $\Delta_{n,N}\setminus\Delta_\infty = \bigsqcup_{\alpha\in\mathcal{A}_0'}\Gamma_\infty\alpha\Gamma_\infty$. With this at hand we can use (10) again to estimate

$$\begin{aligned}
\sum_{\alpha\in\mathcal{A}_0} |j(\alpha,z)|^{-k} &\leq \sum_{\alpha\in\mathcal{A}_0'} \sum_{\gamma\in\Gamma_\infty} |j(\alpha\gamma,z)|^{-k} \\
&= 2 \sum_{\substack{\alpha\in\mathcal{A}_0',\\ \alpha=\begin{pmatrix} * & * \\ c & d\end{pmatrix}}} \sum_{\gamma\in\Gamma_\infty} |c(z+m)+d|^{-k} \\
&\ll_k \operatorname{Im}(z)^{-k}(1+\operatorname{Im}(z)) \sum_{\substack{\alpha\in\mathcal{A}_0',\\ \alpha=\begin{pmatrix} * & * \\ c & *\end{pmatrix}}} |c|^{-k}.
\end{aligned}$$

Thus, according to (9) we find

$$S(z) \ll_k \operatorname{Im}(z)^{-k}(1+\operatorname{Im}(z))^2.$$

This estimate implies convergence and integrability, so that the proof is complete. □

Next we deal with the $\Delta_{\frac{p}{q}}$-part of the inner sum, which we excluded above.

Lemma 4.19. *For $\sigma_{\frac{p}{q}}\infty = \frac{p}{q}$ we have*

$$\begin{aligned}
&\int_{F_{\frac{p}{q}}} \sum_{\alpha\in\Delta_{\frac{p}{q}}} K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k d\mu(z) \\
&\qquad = \lim_{s\to 0}\sum_{\alpha\in\Delta_{\frac{p}{q}}} \int_{F_{\frac{p}{q}}} K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^{k-s}|j(\sigma_{\frac{p}{q}}^{-1},z)|^{2s} d\mu(z).
\end{aligned}$$

Proof. For convenience we focus on $\frac{p}{q} = \infty$. The general case follows by standard reduction arguments.

Fix $\overline{\mathcal{A}}_0$ such that $\Delta_\infty = \bigsqcup_{\alpha \in \overline{\mathcal{A}}_0} \Gamma_\infty \alpha$. Note that one can arrange that $\overline{\mathcal{A}}_0 = \mathcal{A} \cap \Delta_\infty$ with $\mathcal{A}$ as in the previous proof. Further $\overline{\mathcal{A}}_0$ is finite and all its entries have vanishing lower left entry. Following the proof of the last lemma yields

$$\begin{aligned} S_s(z) &= K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^{k-s} \\ &\ll_k (\operatorname{Im}(z) + 1) \operatorname{Im}(z)^{-s} \sum_{\alpha \in \overline{\mathcal{A}}_0} |j(\alpha, z)|^{-k} \ll_{k,n,N} (\operatorname{Im}(z) + 1) \operatorname{Im}(z)^{-s}. \end{aligned}$$

Thus, $S_s(z)$ converges for $z \in F_\infty$ and for $s > 0$ it is integrable on F_∞. This yields to

$$\begin{aligned} &\int_{F_\infty} \sum_{\alpha \in \Delta_\infty} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^{k-s} d\mu(z) \\ &\qquad = \sum_{\alpha \in \Delta_\infty} \int_{F_\infty} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^{k-s} d\mu(z). \end{aligned}$$

The statement, for $\frac{p}{q} = \infty$, follows by taking $s \to 0$ through a monotone decreasing sequence and applying the monotone convergence theorem. □

Remark 4.20. In the following we always understand the limit $s \to 0$ through some monotone decreasing sequence $s_n \to 0$ as $n \to \infty$.

Finally, we treat the bulk.

Lemma 4.21. *We have*

$$\begin{aligned} &\int_{F^\circ} \sum_{\alpha \in \Delta_{n,N}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^{k} d\mu(z) \\ &\qquad = \sum_{\alpha \in \Delta_{n,N}} \int_{F^\circ} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^{k} d\mu(z). \end{aligned}$$

Proof. This follows easily from the fact that F° is compact and the convergence properties of $K_{k,N}$. □

Now we want to combine the three pieces again. To do so we put

$$\Delta_2 = \bigcup_{\frac{p}{q}} \Delta_{\frac{p}{q}} \setminus Z \text{ and } \Delta_1 = \Delta_{n,N} \setminus \Delta_2.$$

Lemma 4.22. *The sets Δ_1, Δ_2 are stable under conjugation by $\Gamma_0(N)$.*

Proof. This follows after observing that for $\gamma \in \Gamma_0(N)$ we have $\gamma^{-1} z \gamma = z$ for all $z \in Z$ and

$$\Delta_{\gamma^{-1} \frac{p}{q}} = \gamma^{-1} \Delta_{\frac{p}{q}} \gamma.$$

We leave the details as an exercise. □

In particular the sets $\mathrm{conj}_{\Gamma_0(N)}(\Delta_i)$ of $\Gamma_0(N)$-conjugacy classes make sense. Recall the groups

$$Z(\alpha) = \{\beta \in \mathrm{GL}_2^+(\mathbb{Q})\colon \alpha\beta = \beta\alpha\} \text{ and } \Gamma(\alpha) = \{\gamma \in \Gamma_0(N)\colon \gamma^{-1}\alpha\gamma = \alpha\}.$$

With these notions at hand we can formulate the trace formula in its coarse form.

Proposition 4.23. *In the notation as above we have*

$$\begin{aligned}\mathrm{Tr}(T_n) = \frac{n^{k-1}}{2} \sum_{\alpha \in conj_{\Gamma_0(N)}(\Delta_1)} \int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k d\mu(z) \\ + \frac{n^{k-1}}{2} \lim_{s\to 0} \sum_{\alpha \in conj_{\Gamma_0(N)}(\Delta_2)} \int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k f_s(z,\alpha) d\mu(z),\end{aligned}$$

for

$$f_s(z,\alpha) = \begin{cases} \operatorname{Im}(z)^{-s} |j(\sigma_{\frac{p}{q}}, z)|^{2s} & \text{if } z \in U_{\frac{p}{q}} \text{ and } \alpha\frac{p}{q} = \frac{p}{q} \text{ for some cusp } \frac{p}{q}, \\ 1 & \text{else.} \end{cases}$$

Proof. We first observe that $F_{\frac{p}{q}} = \emptyset$ for all but finitely many cusps. Thus, we can write $\mathcal{F}_N = F^\circ \sqcup \bigsqcup_{j=1}^l F_{\frac{p_j}{q_j}}$. Splitting the integral and the sum in Proposition 4.17 and applying Lemma 4.18,4.19 and 4.21 yields

$$\mathrm{Tr}(T_n) = \frac{n^{k-1}}{2} \lim_{s\to 0} \sum_{\alpha\in\Delta_{n,N}} \int_{\mathcal{F}_N} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k f_s(z,\alpha) d\mu(z).$$

Next we split the α-sum in a sum over $\alpha \in \Delta_1$ and a sum over $\alpha \in \Delta_2$. One checks, that it is save to take the limit inside the Δ_1-sum and inside the integral. Thus, we arrive at

$$\begin{aligned}\mathrm{Tr}(T_n) = \frac{n^{k-1}}{2} \sum_{\alpha\in\Delta_1} \int_{\mathcal{F}_N} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k d\mu(z) \\ + \frac{n^{k-1}}{2} \lim_{s\to 0} \sum_{\alpha\in\Delta_2} \int_{\mathcal{F}_N} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k f_s(z,\alpha) d\mu(z),\end{aligned}$$

It remains to arrange the sums in conjugacy classes. We start by writing

$$\Delta_1 = \bigsqcup_{\alpha\in \mathrm{conj}_{\Gamma_0(N)}(\Delta_1)} \bigsqcup_{\gamma\in\Gamma(\alpha)\backslash\Gamma_0(N)} \{\gamma^{-1}\alpha\gamma\}.$$

Now we can compute

$$\sum_{\alpha\in\Delta_1}\int_{\mathcal{F}_N} K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k d\mu(z)$$
$$= \sum_{\alpha\in\operatorname{conj}_{\Gamma_0(N)}(\Delta_1)}\ \sum_{\gamma\in\Gamma(\alpha)\backslash\Gamma_0(N)}\int_{\gamma\mathcal{F}_N} K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k d\mu(z).$$

Note that the integrand is $\Gamma(\alpha)$-invariant, so that we have

$$\sum_{\gamma\in\Gamma(\alpha)\backslash\Gamma_0(N)}\int_{\gamma\mathcal{F}_N} K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k d\mu(z) = \int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k d\mu(z)$$

as desired. Running a similar argument for the Δ_2-sum completes the proof. □

The next step is to consider different conjugacy classes of elements separately and compute their contribution explicitly. We distinguish the following sets

$$\Delta_{n,N}^{(e)} = \{\alpha\in\Delta_{n,N}\colon \text{ elliptic}\},\ \Delta_{n,N}^{(p)} = \{\alpha\in\Delta_{n,N}\colon \text{ parabolic}\},$$
$$\Delta_{n,N}^{(h_1)} = \{\alpha\in\Delta_{n,N}\colon \text{ hyperbolic with fixed points in } \mathbb{R}\setminus\mathbb{Q}\} \text{ and}$$
$$\Delta_{n,N}^{(h_2)} = \{\alpha\in\Delta_{n,N}\colon \text{ hyperbolic with fixed points in } \mathbb{Q}\cup\{\infty\}\}.$$

Note that $\Delta_1 = (Z\cap\Delta_{n,N})\cup\Delta_{n,N}^{(e)}\cup\Delta_{n,N}^{(h_1)}$ and $\Delta_2 = \Delta_{n,N}^{(h)}\cup\Delta_{n,N}^{(h_2)}$.

4.4. **The scalar term.** Note that $\alpha = \operatorname{diag}(z,z)\in\Delta_{n,N}$ if and only if $n=m^2$ and $z=\pm m$. Given the explicit form of the Bergman-kernel the computation is almost trivial and we obtain the following result.

Lemma 4.24. *The contribution of the centre to the trace formula is given by*

$$\frac{n^{k-1}}{2}\sum_{\alpha\in Z\cap\Delta_{n,N}}\int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k d\mu(z) = \delta_{n=\square}n^{\frac{k}{2}-1}\frac{k-1}{12}N\psi(N),$$

for $\psi(N)=\prod_{p\mid N}(1+p^{-1})$.

4.5. **The elliptic contribution.** Suppose $\alpha\in\Delta_{n,N}^{(e)}$ is elliptic. Thus there is a unique fixed point $z_0\in\mathbb{H}$ and α fixes also the conjugate $\overline{z_0}$. Furthermore α is diagonalisable with eigenvalues $\lambda,\overline{\lambda}$. More precisely, for $\sigma=\begin{pmatrix}1 & -z_0\\ 1 & -\overline{z_0}\end{pmatrix}$ we have

$$\alpha = \sigma^{-1}\begin{pmatrix}\lambda & 0\\ 0 & \overline{\lambda}\end{pmatrix}\sigma = \frac{1}{z_0-\overline{z_0}}\begin{pmatrix}\overline{\lambda}z_0-\lambda\overline{z_0} & |z_0|^2(\lambda-\overline{\lambda})\\ \overline{\lambda}-\lambda & \lambda z_0-\overline{\lambda}\overline{z_0}\end{pmatrix}.$$

Lemma 4.25. *Let α, σ and λ be as above and put $w=\sigma z$. Then we have*

$$K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k = n^{-k}\frac{k-1}{4\pi}\lambda^k\left(\frac{1-|w|^2}{1-\frac{\lambda}{\overline{\lambda}}|w|^2}\right)^k. \tag{11}$$

Proof. We start by noting that

$$\beta a-\beta b=\frac{\det(\beta)(a-b)}{j(\beta,a)j(\beta,b)}.$$

Put $w'=\sigma\overline{z}$ and note that $w'=\frac{w}{|w|^2}$ and $\lambda\overline{\lambda}=\det(\alpha)$. Using (11) we compute

$$\frac{w-w'}{\frac{\lambda}{\overline{\lambda}}w-w'}=\frac{\sigma z-\sigma\overline{z}}{\sigma\alpha z-\sigma\overline{z}}=\frac{(z-\overline{z}_0)(\alpha z-\overline{z}_0)}{(\alpha z-\overline{z})(z-\overline{z}_0)} \tag{12}$$

Furthermore,

$$(\alpha z-\overline{z}_0)j(\alpha,z)=(\alpha z-\overline{z}_0)j(\sigma^{-1},\mathrm{diag}(\lambda,\overline{\lambda})\sigma z)j(\mathrm{diag}(\lambda,\overline{\lambda})\sigma,z)=\overline{\lambda}(z-\overline{z}_0).$$

With these preliminaries sorted we can check

$$\begin{aligned}K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k&=\frac{k-1}{4\pi}(2i\operatorname{Im}z)^k[(\alpha z-\overline{z})j(\alpha,z)]^{-k}\\&=\frac{k-1}{4\pi}\left[\frac{z-\overline{z}}{\alpha z-\overline{z}}\cdot\frac{\alpha z-\overline{z}_0}{\overline{\lambda}(z-\overline{z}_0)}\right]^k\\&=\frac{k-1}{4\pi}\overline{\lambda}^{-k}\left[\frac{w-w'}{\frac{\lambda}{\overline{\lambda}}w-w'}\right]^k.\end{aligned}$$

This finishes the proof. □

Next note that $\Gamma(\alpha)=\Gamma_0(N)_{z_0}$. Furthermore, the stabiliser $\Gamma_0(N)_{z_0}$ in $\Gamma_0(N)$ is a finite group and almost every $\Gamma_0(N)_{z_0}$-orbit in $\mathbb{H}$ as exactly $\sharp\Gamma_0(N)_{z_0}/\{\pm1\}$ elements. Thus we find

$$\begin{aligned}&\int_{\Gamma(\alpha)\backslash\mathbb{H}}K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^kd\mu(z)\\&\qquad=n^{-k}\frac{k-1}{4\pi}\lambda^k\frac{1}{\sharp\Gamma_0(N)_{z_0}/\{\pm1\}}\int_{\mathbb{H}}\left(\frac{1-|\sigma z|^2}{1-\frac{\lambda}{\overline{\lambda}}|\sigma z|^2}\right)^kd\mu(z).\end{aligned}$$

The remaining integral can be changed by a change ov variables. Note that the transformation σ identifies the upper half plane model of the hyperbolic plane with the Poincare disc model. In other words, $\sigma\mathbb{H}=D$. Thus we compute

$$\begin{aligned}\int_{\mathbb{H}}\left(\frac{1-|\sigma z|^2}{1-\frac{\lambda}{\overline{\lambda}}|\sigma z|^2}\right)^kd\mu(z)&=4\int_D\frac{(1-|w|^2)^{k-2}}{(1-\frac{\lambda}{\overline{\lambda}}|w|^2)^k}d\nu(w)\\&=8\pi\int_0^1\frac{(1-r^2)^{k-2}}{(1-\frac{\lambda}{\overline{\lambda}}r^2)^k}rdr\\&=\frac{4\pi}{k-1}\frac{\overline{\lambda}}{\lambda-\overline{\lambda}}.\end{aligned}$$

We have thus sketched the proof of the following Lemma.

Lemma 4.26. *In the notation as above we have*

$$\frac{n^{k-1}}{2}\int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k d\mu(z) = \frac{1}{\sharp\Gamma_0(N)_{z_0}}\cdot\frac{\lambda^{k-1}}{\overline{\lambda}-\lambda}.$$

Proposition 4.27. *The elliptic contribution is given by*

$$\frac{n^{k-1}}{2}\sum_{\alpha\in conj_{\Gamma_0(N)}\Delta_{n,N}^{(e)}}\int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k d\mu(z)$$

$$= -\frac{1}{2}\sum_{\substack{t\in\mathbb{Z},\\ t^2-4n<0}}\frac{\lambda_n(t)^{k-1}-\overline{\lambda_t(n)}^{k-1}}{\lambda_n(t)-\overline{\lambda_t(n)}}B_n(t),$$

where $\lambda_t(n)$ *and* $\overline{\lambda_t(n)}$ *are the two solutions of the polynomial* X^2-tX+n *and*

$$B_n(t) = \sum_{\substack{\alpha\in conj_{\Gamma_0(N)}\Delta_{n,N}^{(e)},\\ Tr(\alpha)=t}}\frac{1}{\sharp\Gamma(\alpha)}.$$

Proof. First, let α be an elliptic element. Then $\alpha'=a(-1)^{-1}\alpha a(-1)$ is also an elliptic element in $\Delta_{n,N}$. Note that they are conjugate with respect to $\mathrm{GL}_2(\mathbb{Z})$ but not $\Gamma_0(N)$. Thus they represent two different conjugacy classes but have the same trace (and determinant). By grouping all such α and α' together Lemma 4.26 yields

$$\frac{n^{k-1}}{2}\sum_{\alpha\in\mathrm{conj}_{\Gamma_0(N)}\Delta_{n,N}^{(e)}}\int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k d\mu(z)$$

$$= \frac{1}{2}\sum_{\alpha\in\mathrm{conj}_{\Gamma_0(N)}\Delta_{n,N}^{(e)}}\frac{1}{\sharp\Gamma(\alpha)}\frac{\lambda_\alpha^{k-1}-\overline{\lambda_\alpha}^{k-1}}{\lambda_\alpha-\overline{\lambda_\alpha}}.$$

The statement now follows by arranging conjugacy classes according to their trace. □

Now we make some more algebraic definitions. Given an integer $d\equiv 0,1 \bmod 4$ with $d<0$, there is a unique order S_d of discriminant d in the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$. Let $h(d)=h(S_d)$ denote the (narrow) class number of S_d. We also modify the Kronecker-symbol as follows. Given a fundamental discriminant[5] d and

[5]A fundamental discriminant are exactly those integers that appear as discriminants of quadratic fields. In other words, d is either square-free and $d\equiv 1 \mod 4$ or $\frac{d}{4}$ is square free and satisfies $\frac{d}{4}\equiv 2,3 \mod 4$.

an integer m we define

$$\left\{\frac{dm^2}{p}\right\} = \begin{cases} 1 & \text{if } p \mid m, \\ \left(\frac{d}{p}\right) & \text{else.} \end{cases} \tag{13}$$

Lemma 4.28. *Suppose N is square-free and $(n, N) = 1$, then*

$$B_n(t) = \sum_{f^2 \mid t^2-4n} \frac{h\left(\frac{t^2-4n}{f^2}\right)}{\sharp(S_d^\times/\{\pm 1\})} \cdot \prod_{p \mid N} \left(1 + \left\{\frac{(t^2-4n)/f^2}{p}\right\}\right),$$

for all t with $t^2 - 4n < 0$.

Proof. We start by some more general considerations. Let $R = R(N)$ be the ring of all integer 2×2-matrices with lower left entry divisible by N. In particular, $\Delta_{n,N} \subset R$. Given any $\alpha \in R$ we consider the full $\mathrm{GL}_2(\mathbb{Q})$-orbit

$$C(\alpha) = \{\delta\alpha\delta^{-1} \colon \delta \in \mathrm{GL}_2(\mathbb{Q})\}.$$

One checks that for $\alpha \in R$ we have $\Delta_{n,N}^{(e)} \cap C(\alpha) = \Delta_{n,N} \cap C(\alpha)$.

If $\alpha, \beta \in \Delta_{n,N}^{(e)}$ have the same trace, then they are conjugate by an element in $\mathrm{GL}_2(\mathbb{Q})$ (not necessarily by an element of $\mathrm{GL}_2(\mathbb{Z})$). Thus given any $\alpha \in M_2(\mathbb{Q})$ with determinant n and trace t we can write

$$B_n(t) = \sum_{\beta \in \mathrm{conj}_{\Gamma_0(N)}[\Delta_{n,N} \cap C(\alpha)]} \frac{1}{\sharp\Gamma(\beta)}.$$

Sine α is elliptic we have that $\mathbb{Q}[\alpha] \cong \mathbb{Q}[X]/(f(X))$, for $f(X) = X^2 - tX + n$, is an imaginary quadratic field of discriminant $D_{t,n} = \frac{t^2-4n}{m^2}$, where m^2 is essentially the square-part of $t^2 - 4n$. In particular $D_{t,n} \equiv 0, 1 \bmod 4$ and $D_{t,n} < 0$. In other words, we have $\mathbb{Q}[\alpha] \cong \mathbb{Q}(\sqrt{D_{t,n}})$. For each $f \in \mathbb{N}$ there is a unique order $\mathfrak{r}_f \subset \mathbb{Q}[\alpha]$ of discriminant $D_{t,n}f^2$. We decompose

$$C(\alpha) = \bigsqcup_{f \in \mathbb{N}} C(\alpha, f), \text{ for } C(\alpha, f) = \{\delta\alpha\delta^{-1} \colon \mathbb{Q}[\alpha] \cap \delta^{-1}R\delta = \mathfrak{r}_f\}.$$

We claim that for $\beta \in C(\alpha, f)$ one has $\sharp\Gamma(\beta) = \sharp\mathfrak{r}_f^\times$ and leave the proof as an exercise.[6] Further, if $\Delta_{n,N} \cap C(\alpha, f) \neq \emptyset$, then $\mathfrak{r}_f \supset \mathbb{Z}[\alpha]$.[7] In other words, $f \mid m$ and in particular $f^2 \mid [t^2 - 4n]$. Thus, we can write

$$B_n(t) = \sum_{f^2 \mid t^2-4n} \frac{1}{\sharp\mathfrak{r}_f^\times} \sharp\mathrm{conj}_{\Gamma_0(N)}[\Delta_{n,N} \cap C(\alpha, f)].$$

---

[6]This follows by observing that

$$\Gamma(\beta) = \{\gamma \in \Gamma_0(N) \colon \gamma\beta = \beta\gamma\} = \mathbb{Q}[\beta] \cap R^\times = \delta(\mathbb{Q}[\alpha] \cap \delta^{-1}R^\times\delta)\delta^{-1} = \delta\mathfrak{r}_f^\times\delta^{-1}.$$

[7]This follows since $\alpha \in \delta^{-1}R\delta \cap \mathbb{Q}[\alpha] = \mathfrak{r}_f$ for δ such that $\beta = \delta\alpha\delta^{-1} \in \Delta_{n,N} \cap C(\alpha, \mathfrak{r}_f)$.

We now have to take a leap of faith and believe that the numbers

$$\sharp\mathrm{conj}_{\Gamma_0(N)}(\Delta_{n,N} \cap C(\alpha, f))$$

can be computed precisely. This can be achieved using a local to global principle, but going into details would take us to far afield. Furthermore, we will encounter (essentially) the same problem later on when we are dealing with quaternion algebras and we will say more then. Indeed one can show that

$$\sharp\mathrm{conj}_{\Gamma_0(N)}[\Delta_{n,N} \cap C(\alpha, f)] \\ = 2h(\mathfrak{r}_f) \prod_{p\mid N} \sharp\mathrm{conj}_{\mathcal{O}_p^\times}\{\delta^{-1}\alpha\delta \in \mathcal{O}_p \colon \mathbb{Q}_p[\alpha] \cap \delta^{-1}R_p\delta = [\mathfrak{r}_f]_p\},$$

where $[\mathfrak{r}_f]_p$ and R_p are the localisations of R and $\mathfrak{r}_f$ and

$$\mathcal{O}_p = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \colon c \in N\mathbb{Z}_p,\ a \in \mathbb{Z}_p^\times,\ \det(\gamma) \neq 0)\right\}$$

The cardinality of the local sets can be computed yields precisely the numbers claimed in the lemma. □

Exercise 6. Find two (elliptic) matrices $\alpha, \beta \in \mathrm{SL}_2(\mathbb{Z})$, which are conjugate in $\mathrm{GL}_2(\mathbb{Q})$ but not in $\mathrm{SL}_2(\mathbb{Z})$.

4.6. **The hyperbolic contribution.** Let α be a hyperbolic element with distinct fixed points $x_1, x_2 \in \mathbb{R} \cup \{\infty\}$. Without loss of generality we assume $x_2 > x_1$. We put

$$\sigma = (x_1 - x_2)^{-\frac{1}{2}} \begin{pmatrix} 1 & -x_1 \\ 1 & -x_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Similar to the elliptic case we find

$$\alpha = \sigma^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \sigma = \frac{1}{x_2 - x_1} \begin{pmatrix} \lambda_2 x_2 - \lambda - 1x_1 & x_1x_2(\lambda_1 - \lambda_2) \\ \lambda - 2 - \lambda_1 & \lambda - 1x_2 - \lambda_2 x_1 \end{pmatrix},$$

for the two distinct (real) eigenvalues λ_1, λ_2 of α.

We put $w = \sigma z$ and $w' = \sigma\overline{z} = \overline{w}$. One finds (compare Lemma 4.25)

$$K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k = \frac{k-1}{4\pi} \lambda_2^{-k} \left(\frac{w - \overline{w}}{\frac{\lambda_1}{\lambda_2} w - \overline{w}} \right)^k.$$

From here we have to distinguish two cases.

4.6.1. *Hyperbolic conujugacy classes of type one.* As the title suggests we now assume α to be of type one. In other words $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}$. In this case

$$\Gamma(\alpha) = \Gamma_0(N)_{x_1} \cap \Gamma_0(N)_{x_2}.$$

Lemma 4.29. *There is $u > 1$ such that*

$$\{\pm 1\} \cdot (\sigma\Gamma(\alpha)\sigma^{-1}) = \left\{ \pm \begin{pmatrix} u^m & 0 \\ 0 & u^{-m} \end{pmatrix} : m \in \mathbb{Z} \right\}.$$

Proof. If $\Gamma(\alpha) \not\subset Z$, then the statement follows at once since $\sigma\Gamma(\alpha)\sigma^{-1}$ stabilises ∞ and 0. It remains to exclude $\Gamma(\alpha) \not\subset Z$. However, if that would be true then $\Gamma(\alpha)\backslash\mathbb{H} = \mathbb{H}$. But using elementary computations one can show that

$$\int_{\Gamma(\alpha)\backslash\mathbb{H}} |K_k(\alpha z, z)j(\alpha, z)^{-k}\operatorname{Im}(z)^k|d\mu(z) = \frac{k-1}{4\pi}\lambda_2^{-k}\int_0^\infty \frac{1}{r}dr\int_0^\pi \frac{1}{\sin(\varphi)^2}\left|\frac{e^{i\varphi}-e^{-i\varphi}}{\frac{\lambda_1}{\lambda_2}e^{i\varphi}-e^{-i\varphi}}\right|^k d\varphi,$$

which is a contradiction since the right hand side does not converge. □

As a fundamental domain of $\sigma\Gamma(\alpha)\sigma^{-1}\backslash\mathbb{H}$ we can choose $\{w \in \mathbb{H} \colon 1 \leq |w| < u^2\}$. Turning to the orbital integral we find:

$$\begin{aligned}
&\int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z, z)j(\alpha, z)^{-k}\operatorname{Im}(z)^k d\mu(z)\\
&= \frac{k-1}{4\pi}\lambda_2^{-k}\int_1^{u^2}\frac{1}{r}dr\int_0^\pi\frac{1}{\sin(\varphi)^2}\left(\frac{e^{i\varphi}-e^{-i\varphi}}{\frac{\lambda_1}{\lambda_2}e^{i\varphi}-e^{-i\varphi}}\right)^k d\varphi\\
&= -2\ln(u)\frac{k-1}{\pi}\lambda_2^{-k}\int_0^\pi\frac{(e^{i\varphi}-e^{-i\varphi})^{k-2}}{(\frac{\lambda_1}{\lambda_2}e^{i\varphi}-e^{-i\varphi})^k}d\varphi.
\end{aligned}$$

Lemma 4.30. *For $1 \neq \lambda_1/\lambda_1 > 0$ we have*

$$\int_0^\pi \frac{(e^{i\varphi}-e^{-i\varphi})^{k-2}}{(\frac{\lambda_1}{\lambda_2}e^{i\varphi}-e^{-i\varphi})^k}d\varphi = 0.$$

In particular hyperbolic elements of type one do not contribute to the trace formula.

Proof. We set $\lambda = \frac{\lambda_1}{\lambda_2}$ and denote the integrand by $f_\lambda(\varphi)$. Note that $\lambda \neq 0, 1$, since $\lambda_1 \neq \lambda_2$. Since k is even $f_\lambda(\varphi+\pi) = f_\lambda(\varphi)$. Thus we have

$$\int_0^\pi f_\lambda(\varphi)d\varphi = \frac{1}{2}\int_0^{2\pi} f_\lambda(\varphi)d\varphi = -\frac{i}{2}\int_{S_1}\frac{(z^2-1)^{k-2}}{(\lambda z^2-1)^k}zdz.$$

If $\lambda < 1$, then the integrand of the latter contour integral is holomorphic in a neighbourhood of $B_1(0)$. Thus the integral vanishes.

In the case $\lambda > 1$ we compute the integral via the residual theorem. This yields

$$\int_0^\pi f_\lambda(\varphi)d\varphi = \pi\sum_\pm \operatorname{res}_{z=\pm\lambda^{-\frac{1}{2}}}\left[\frac{(z^2-1)^{k-2}}{(\lambda z^2-1)^k}z\right] = \frac{\pi}{(k-1)!}\lambda^{-\frac{k}{2}}\sum_\pm\frac{d^{k-1}}{dz^{k-1}}g_\pm(\pm\lambda^{-\frac{1}{2}}),$$

for $g_\pm(z) = z\frac{(z^2-1)^{k-2}}{(\sqrt{\lambda}z\mp 1)}$. Since $g_\pm(-z) = g_\mp(z)$ and $k-1$ is odd we see that the two residues cancel each other out. □

4.6.2. *Hyperbolic conjugacy classes of type two.* Now α is hyperbolic of type two. In other words $x_1, x_2 \in \mathbb{Q}\cup\{\infty\}$. In this case $\Gamma(\alpha) = \{\pm 1\}$ and $\Gamma(\alpha)\backslash\mathbb{H} = \mathbb{H}$. We now have to compute the integral

$$\int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k f_s(z, \alpha) d\mu(z),$$

for

$$f_s(z,\alpha) = \begin{cases} \operatorname{Im}(z)^{-s} |j(\sigma_{\frac{p}{q}}, z)|^{2s} & \text{if } z \in U_{\frac{p}{q}} \text{ and } \alpha\frac{p}{q} = \frac{p}{q} \text{ for some cusp } \frac{p}{q}, \\ 1 & \text{else.} \end{cases}$$

Take σ_i such that $\sigma_i\infty = x_1$ for $i = 1, 2$. Further define $U_i = \sigma_i U_\infty$.

Lemma 4.31. *For $i = 1, 2$ we have*

$$\int_{U_i} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k \operatorname{Im}(z)^{-s} |j(\sigma_i^{-1}, z)|^{2s} d\mu(z) = 0.$$

Proof. Let us treat the case $i = 1$. Note that $\sigma\sigma_1\infty = \infty$, so that $\sigma\sigma_1 = \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}$. In particular $|j((\sigma\sigma_1)^{-1}, w)| = |a|$ and

$$\sigma U_1 = (\sigma\sigma_1)U_\infty = \{z \in \mathbb{H}\colon \operatorname{Im}(z) > a^2\delta\}.$$

We compute

$$\begin{aligned}
&\int_{U_1} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k \operatorname{Im}(z)^{-s} |j(\sigma_1^{-1}, z)|^{2s} d\mu(z) \\
&= \frac{k-1}{4\pi}\lambda_2^{-k} \int_{\sigma U_1} \operatorname{Im}(w)^{-s} \left(\frac{w - \overline{w}}{\frac{\lambda_1}{\lambda_2} w - \overline{w}} \right)^k |j((\sigma\sigma_1)^{-1}, w)|^{2s} d\mu(w) \\
&= -\frac{k-1}{\pi}\lambda_2^{-k}|a|^{2s} \int_0^\pi \frac{(e^{i\varphi} - e^{-i\varphi})^{k-2}}{(\frac{\lambda_1}{\lambda_2}e^{i\varphi} - e^{-i\varphi})^k} \sin(\varphi)^{-s} \int_{\frac{a^2\delta}{\sin(\varphi)}}^\infty r^{-s-1} dr d\varphi \\
&= -\frac{k-1}{s\pi}\lambda_2^{-k}\delta^{-s} \int_0^\pi \frac{(e^{i\varphi} - e^{-i\varphi})^{k-2}}{(\frac{\lambda_1}{\lambda_2}e^{i\varphi} - e^{-i\varphi})^k} d\varphi.
\end{aligned}$$

Note that we have seen that the remaining φ-integral vanishes.

For $i = 2$ the situation is similar. The difference being that $\sigma\sigma_2\infty = 0$, so that $\sigma\sigma_2$ is of the form $\begin{pmatrix} 0 & b \\ b^{-1} & * \end{pmatrix}$. we leave the details to the reader. $\square$

It remains to compute the integral over the bulk $H' = \mathbb{H} \setminus (U_1 \cup U_2)$. Note that

$$\sigma H' = \{re^{i\varphi}\colon \frac{b^2\sin(\varphi)}{\delta} \le r \le \frac{a^2\delta}{\sin(\varphi)}\}.$$

The usual steps lead to

$$\begin{aligned}
&\int_{H'} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k d\mu(z) \\
&= -\frac{k-1}{\pi}\lambda_2^{-k}\int_0^\pi \frac{(e^{i\varphi}-e^{-i\varphi})^{k-2}}{(\frac{\lambda_1}{\lambda_2}e^{i\varphi}-e^{-i\varphi})^k}\int_{\frac{b^2\sin(\varphi)}{\delta}}^{\frac{a^2\delta}{\sin(\varphi)}} r^{-1} dr d\varphi \\
&= -2\frac{k-1}{\pi}\lambda_2^{-k}\int_0^\pi \frac{(e^{i\varphi}-e^{-i\varphi})^{k-2}}{(\frac{\lambda_1}{\lambda_2}e^{i\varphi}-e^{-i\varphi})^k}\left[\ln(\frac{a\delta}{b}) - \ln(\sin(\varphi))\right] d\varphi \\
&= 2\frac{k-1}{\pi}\lambda_2^{-k}\int_0^\pi \frac{(e^{i\varphi}-e^{-i\varphi})^{k-2}}{(\frac{\lambda_1}{\lambda_2}e^{i\varphi}-e^{-i\varphi})^k}\ln(\sin(\varphi))d\varphi.
\end{aligned}$$

Note that the remaining integral is independent of a, b and δ as well as s.

Lemma 4.32. *We have*

$$I = \int_0^\pi \frac{(e^{i\varphi}-e^{-i\varphi})^{k-2}}{(\frac{\lambda_1}{\lambda_2}e^{i\varphi}-e^{-i\varphi})^k}\ln(\sin(\varphi))d\varphi = \frac{\pi}{2(k-1)}\cdot\begin{cases}\frac{\lambda_2}{\lambda_1-\lambda_2} & \text{for } |\lambda_1|>|\lambda_2|, \\ -\frac{\lambda_2^k\lambda_1^{1-k}}{\lambda_1-\lambda_2} & \text{for } |\lambda_1|<|\lambda_2|.\end{cases}$$

Proof. As before we put $\lambda = \lambda_1/\lambda_2 \neq 0$. Since $\lambda_1\lambda_2 = \det(\alpha) > 0$ we conclude that $\lambda > 0$, as λ_1 and λ_2 must have the same sign. First one computes that

$$\frac{(e^{i\varphi}-e^{-i\varphi})^{k-2}}{(\lambda e^{i\varphi}-e^{-i\varphi})^k} = \frac{1}{2i(k-1)(\lambda-1)}\frac{d}{d\varphi}\left[\left(\frac{e^{i\varphi}-e^{-i\varphi}}{\lambda e^{i\varphi}-e^{-i\varphi}}\right)^{k-1}\right].$$

Integration by parts yields

$$\begin{aligned}
I &= -\frac{(2i)^{k-2}}{(k-1)(\lambda-1)}\int_0^\pi (\lambda e^{i\varphi}-e^{-i\varphi})^{1-k}\sin(\varphi)^{k-2}\cos(\varphi)d\varphi \\
&= \frac{i}{4(k-1)(\lambda-1)}\int_{S_1}\frac{(z^2-1)^{k-2}(z^2+1)}{(\lambda z^2-1)^{k-1}z}dz.
\end{aligned}$$

This can now be evaluated using the residual theorem once again. If $\lambda < 1$, then the only pole in the unite disc is at $z = 0$. In this case one simply gets

$$I = \frac{\pi}{2(k-1)(\lambda-1)}.$$

To deal with the remaining case we can avoid computing the extra residues by making same elementary manipulations in the beginning to switch $\lambda \rightsquigarrow \frac{1}{\lambda}$. $\square$

We summarise the our findings in the following proposition.

Proposition 4.33. *The hyperbolic contribution of second type is given by*

$$\frac{n^{k-1}}{2}\lim_{s\to 0}\sum_{\alpha\in conj_{\Gamma_0(N)}(\Delta_2)^{(h_2)}}\int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k f_s(z,\alpha)d\mu(z)$$
$$= -\frac{1}{2}\sum_{\substack{t\in\mathbb{Z},\\ t^2-4n>0}}\frac{\min(|\lambda_n^+(t)|,|\lambda_n^-(t)|)^{k-1}}{|\lambda_n^+(t)-\lambda_n^-(t)|}C_n(t),$$

where $\lambda_t^+(n)$ and $\lambda_t^-(n)$ are the two solutions of the polynomial X^2-tX+n and

$$C_n(t)=\sum_{\substack{\alpha\in conj_{\Gamma_0(N)}\Delta_{n,N}^{(h_2)},\\ Tr(\alpha)=t}} 1.$$

Proof. The statement follows from the considerations above and arranging conjugacy classes according to their trace. □

Lemma 4.34. *Suppose N is squarefree and $(n,N)=1$,*

$$C_n(t)=\delta_{t^2-4n=\square}2^{\omega(N)}\sum_{f^2|t^2-4n}\phi(f)$$

for all t with $t^2-4n>0$. Here $\phi(f)=f\prod_{p|f}(1-p^{-1})$ is the Euler totient function and $\omega(N)=\sharp\{p\mid N\}$.

Proof. We start by arguing as in Lemma 4.28. In particular for any $\alpha\in R$ with determinant n and trace t we have

$$C_n(t)=\sharp\mathrm{conj}_{\Gamma_0(N)}(C(\alpha)\cap\Delta_{n,N}).$$

The difference is, that since α is hyperbolic of type 2 we have $\mathbb{Q}[\alpha]\equiv\mathbb{Q}\times\mathbb{Q}$. We want to follow the same strategy as in the elliptic case and arrange the elements we are counting by orders. Let us collect some facts to do so. The unique maximal order in $\mathbb{Q}\times\mathbb{Q}$ is $\mathfrak{r}_1=\mathbb{Z}\times\mathbb{Z}$. Again, given $f\in\mathbb{N}$ we have one order $\mathfrak{r}_f$ with index $[\mathfrak{r}_1:\mathfrak{r}_1]=f$. further, note that since α must have fixed points in $\mathbb{Q}\cup\{\infty\}$, the characteristic polynomial must have a square discriminant. Thus we conclude that $t^2-4n=m^2>0$. Further we conclude that $[\mathfrak{r}_1:\mathbb{Z}[\alpha]]=m$.

Borrowing notation (and the argument with the necessary modifications) from the elliptic case we arrive at

$$C_n(t)=\sum_{f|m}\sharp\mathrm{conj}_{\Gamma_0(N)}(\Delta_{n,N}\cap C(\alpha,\mathfrak{r}_f)).$$

This reduces to the local problem (compare to the elliptic case)

$$C_n(t)=\sum_{f|m}\phi(f)\prod_{p|N}\sharp\mathrm{conj}_{\mathcal{O}_p^\times}\{\delta^{-1}\alpha\delta\in\mathcal{O}_p\colon \mathbb{Q}_p[\alpha]\cap\delta^{-1}R_p\delta=[\mathfrak{r}_f]_p\}.$$

Note that in this case the class number is replaced by the Euler totient function, which appears strange but is completely natural from an adelic point of view. Again giving all the details goes beyond the scope of this lecture.

Computing the cardinality of the remaining sets is a completely local task and completing it produces the statement of the lemma. □

4.7. **The parabolic contribution.** Let $\alpha \in \Delta_{n,N}$ be parabolic and let $x \in \mathbb{Q} \cup \{\infty\}$ be its unique fixed point. We write $\mathfrak{a}$ for the equivalence class of cusps x belongs to. Note that there is $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma\infty = x$ Necessarily we find

$$\sigma^{-1}\alpha\sigma = \begin{pmatrix} \lambda & B \\ 0 & \lambda \end{pmatrix},$$

for some $B \in \mathbb{Q}^\times$ and $\lambda \in \mathbb{Q}^\times$ such that $\lambda^2 = n$. Thus, we observe that we can only have parabolic contributions if n is a perfect square.

Note that $\Gamma(\alpha) = \Gamma_0(N)_x$ and

$$\{\pm 1\} \cdot (\sigma^{-1}\Gamma_0(N)_x\sigma) = \left\{ \pm \begin{pmatrix} 1 & hm \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\},$$

where h is the width of the cusp $\mathfrak{a}$. Thus we can choose the fundamental domain

$$\{w \in \mathbb{H}\colon |\mathrm{Re}(w)| \leq \frac{h}{2}\}$$

for $(\sigma^{-1}\Gamma(\alpha)\sigma)\backslash\mathbb{H}$.

Lemma 4.35. *Let σ, B and λ be as above. We have*

$$K_k(\alpha\sigma z, \sigma z) j(\alpha, \sigma z)^{-k} \operatorname{Im}(\sigma z)^k = \frac{k-1}{4\pi}\lambda^{-k} \left(\frac{\operatorname{Im}(z)}{\operatorname{Im}(z) - i\mu} \right)^k$$

for $\mu = \frac{B}{2\lambda}$.

Proof. Left as an exercise. □

We take the following lemma for granted.

Lemma 4.36. *A set of representatives for $conj_{\Gamma_0(N)}(\Delta_{n,N}^{(p)})$ is given by*

$$\bigsqcup_{x \in C(N)} \Delta_x^{(p)},$$

for $\Delta_x^{(p)} = \Delta_{n,N}^{(p)} \cap \Delta_x$.

Write

$$\lim_{s\to 0} \sum_{\alpha\in \mathrm{conj}_{\Gamma_0(N)}(\Delta_2^{(p)})} \int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k f_s(z,\alpha) d\mu(z)$$
$$= \sum_{x\in C(N)} \Bigg[\lim_{s\to 0} \sum_{\alpha\in\Delta_x^{(p)}} \int_{\Gamma(\alpha)\backslash U_x} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k \operatorname{Im}(z)^{-s} |j(\sigma_x, z)|^{2s} d\mu(z)$$
$$+ \sum_{\alpha\in\Delta_x^{(p)}} \int_{\Gamma(\alpha)\backslash(\mathbb{H}\backslash U_x)} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k d\mu(z) \Bigg].$$

Next we artificially introduce an s-limit in the bulk-term in order to combine the two integrals.

Lemma 4.37. *We have*

$$\sum_{\alpha\in\Delta_x^{(p)}} \int_{\Gamma(\alpha)\backslash(\mathbb{H}\backslash U_x)} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k d\mu(z)$$
$$= \lim_{s\to 0} \sum_{\alpha\in\Delta_x^{(p)}} \int_{\Gamma(\alpha)\backslash(\mathbb{H}\backslash U_x)} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k \operatorname{Im}(z)^{-s} |j(\sigma_x, z)|^{2s} d\mu(z).$$

Proof. Inserting the limit must be carefully justified. However, to save chalk we skip the details. □

As a result we find that

$$\lim_{s\to 0} \sum_{\alpha\in \mathrm{conj}_{\Gamma_0(N)}(\Delta_2)^{(p)}} \int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k f_s(z,\alpha) d\mu(z)$$
$$= \sum_{x\in C(N)} \lim_{s\to 0} \sum_{\alpha\in\Delta_x^{(p)}} \int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k \operatorname{Im}(z)^{-s} |j(\sigma_x, z)|^{2s} d\mu(z).$$

The remaining integral can now be computed.

Lemma 4.38. *We have*

$$\int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k \operatorname{Im}(z)^{-s} |j(\sigma_x, z)|^{2s} d\mu(z)$$
$$= \frac{k-1}{4\pi} h_x \lambda_\alpha^{-k} \frac{i^{1+s}}{\mu_\alpha^{1+s}} \frac{\Gamma(s+1)\Gamma(k-s-1)}{\Gamma(k)}.$$

Proof. Using our computations at the beginning of the section we compute

$$\begin{aligned}
&\int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k \operatorname{Im}(z)^{-s} |j(\sigma_x, z)|^{2s} d\mu(z) \\
&= \int_{(\sigma_x^{-1}\Gamma(\alpha)\sigma_x)\backslash\mathbb{H}} K_k(\alpha\sigma_x z, \sigma_x z) j(\alpha, \sigma_x z)^{-k} \operatorname{Im}(\sigma_x z)^k \operatorname{Im}(z)^{-s} d\mu(z) \\
&= \frac{k-1}{4\pi} \lambda_\alpha^{-k} \int_{(\sigma_x^{-1}\Gamma(\alpha)\sigma_x)\backslash\mathbb{H}} (\operatorname{Im}(z) - i\mu_\alpha)^{-k} \operatorname{Im}(z)^{k-2} d\mu(z) \\
&= \frac{k-1}{4\pi} \lambda_\alpha^{-k} h_x \int_0^\infty \frac{y^{k-s-2}}{(y - i\mu_\alpha)^k} dy.
\end{aligned}$$

To simplify notation let us call the y-integral I.

$$I = \frac{i^{2+s}}{\mu_\alpha^{1+s}} \int_0^{\operatorname{sgn}(\mu_\alpha)\infty} \frac{(it)^{k-s-2}}{(1+it)^k} dt = -\frac{i^{1+s}}{\mu_\alpha^{1+s}} \int_\gamma u^s (1-u)^{k-s-2} du,$$

where γ is a path with $\gamma(0) = 1$ and $\gamma(1) = 0$. Note that since the integrand is holomorphic for s small enough the integral does not depend on the choice of γ. Thus we can pick $\gamma \colon r \mapsto 1 - r$. We find

$$I = \frac{i^{1+s}}{\mu_\alpha^{1+s}} \int_{\gamma^{-1}} u^s (1-u)^{k-s-2} du = \frac{i^{1+s}}{\mu_\alpha^{1+s}} \int_0^1 r^s (1-r)^{k-s-2} dr = \frac{i^{1+s}}{\mu_\alpha^{1+s}} B(s+1, k-s-1).$$

The result follows by expressing the beta function as a Γ-quotient. $\square$

For fixed $x \in C(N)$ of width h_x we obtain

$$\begin{aligned}
\lim_{s\to 0} \sum_{\alpha \in \Delta_x^{(p)}} &\int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k \operatorname{Im}(z)^{-s} |j(\sigma_x, z)|^{2s} d\mu(z) \\
&= \frac{h_x n^{-\frac{k}{2}}}{2\pi} \lim_{s\to 0} \sum_{\alpha \in \Delta_x^{(p)}} \operatorname{sgn}(\lambda_\alpha)^k \left(\frac{i\lambda_\alpha}{B_\alpha}\right)^{1+s}.
\end{aligned}$$

Thus we have obtained the following result.

Proposition 4.39. *The parabolic contribution is given by*

$$\begin{aligned}
\frac{n^{k-1}}{2} \lim_{s\to 0} \sum_{\alpha \in conj_{\Gamma_0(N)}(\Delta_2)^{(p)}} &\int_{\Gamma(\alpha)\backslash\mathbb{H}} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k f_s(z, \alpha) d\mu(z) \\
&= \delta_{n=\square} \frac{n^{\frac{k}{2}-1}}{4\pi} \lim_{s\to 0} \sum_{\alpha \in conj_{\Gamma_0(N)}(\Delta_{n,N})^{(p)}} \left(\frac{ih_\alpha \lambda_\alpha}{B_\alpha}\right)^{1+s}.
\end{aligned}$$

We still want to further polish this sum. For the upcoming computations we always assume $n = m^2$, since otherwise the parabolic contribution vanishes. We further set

$$m(\alpha) = \frac{B_\alpha}{h_\alpha \lambda_\alpha}$$

and call the parabolic contribution P. Let us start with the following lemma.

Lemma 4.40. *We have*

$$P = -\delta_{n=\square} \lim_{s\to 0} \frac{n^{\frac{k}{2}-1}}{8} s \sum_{\alpha \in conj_{\Gamma_0(N)}(\Delta^{(p)}_{n,N})} |m(\alpha)|^{-1-s}.$$

Proof. So far we have seen that

$$P = \delta_{n=\square} \frac{n^{\frac{k}{2}-1}}{4\pi} \lim_{s\to 0} \sum_{\alpha \in \mathrm{conj}_{\Gamma_0(N)}(\Delta_{n,N})^{(p)}} (-im(\alpha))^{-1-s}.$$

Let $g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Take $\alpha \in \Delta^{(p)}_{n,N}$ and put $\alpha' = g\alpha g^{-1}$. These two matrices $\Gamma_0(N)$-conjugate. Let x be the fixed point of α and $x' = gx$ the fixed point of α'. Further take $\sigma x = \infty$ and put $\sigma' = g\sigma g^{-1}$. We have

$$\sigma\alpha\sigma^{-1} = \begin{pmatrix} \lambda_\alpha & B_\alpha \\ 0 & \lambda_\alpha \end{pmatrix} \text{ and } \sigma'\alpha\sigma'^{-1} = \begin{pmatrix} \lambda_\alpha & -B_\alpha \\ 0 & \lambda_\alpha \end{pmatrix}.$$

Note that x and x' have the same width. In particular, $m(\alpha') = -m(\alpha)$. The key observation follows from

$$\frac{1}{\pi}(i^{1+s} + (-i)^{1+s}) = -s + O(s^2).$$

One completes the proof by combining the contributions α and α'. Note that some care is needed to argue that the higher order terms can be ignored. We skip the details. $\square$

Proposition 4.41. *For squarefree N and $(n, N) = 1$ the parabolic contribution is given by*

$$P = -\delta_{n=\square} \frac{n^{\frac{k-1}{2}}}{2} 2^{\omega(N)}.$$

Proof. Let $n = m^2$. The idea is as essentially as before. Let us start by noting that over $\mathbb{Q}$ the conjugacy class of a parabolic matrix α is determined by the eigenvalue λ_α. Since $\lambda_\alpha^2 = n$ we must have $\lambda_\alpha = \pm m$. Thus we can set $\alpha_\pm = \pm mn(\frac{\pm}{m})$ and write

$$P = -\frac{n^{\frac{k}{2}-1}}{8} \sum_{\pm} \lim_{s\to 0} s \sum_{\beta \in \mathrm{conj}_{\Gamma_0(N)}(\Delta_{n,N} \cap C(\alpha_\pm))} |m(\beta)|^{1+s}.$$

The trick is again to arrange the remaining conjugacy classes according to their orders. Put $\epsilon = \beta \pm m$ and observe $\epsilon^2 = 0$. Further $\mathbb{Q}[\alpha] = \mathbb{Q}[\epsilon]$ and $\mathbb{Z}[\alpha] = \mathbb{Z}[\epsilon]$. We parametrise all orders by $\mathfrak{r}_f = \mathbb{Z} + \mathbb{Z}\frac{\epsilon}{f}$. In particular $\mathbb{Z}[\alpha] = \mathbf{r}_1 \subset \mathbf{r}_f$ for all $f \in \mathbb{N}$. Further one can proof that, if $\beta \in C(\alpha_\pm, f)$, then

$$|m(\beta)| = 2\frac{l}{m}.$$

With this at hand, we can proceed as previously taking the local to global argument for granted. Note that the analogue of the class number in this case is simply 1. We arrive at

$$\begin{aligned}
P &= -\frac{n^{\frac{k}{2}-1}}{4}\sum_{\pm}\lim_{s\to 0} sm^{1+s}\sum_{f\in\mathbb{N}} f^{-1-s}\prod_{p\mid N}\sharp\mathrm{conj}_{\mathcal{O}_p^\times}\{\delta^{-1}\alpha\delta \in \mathcal{O}_p\colon \mathbb{Q}_p[\alpha]\cap\delta^{-1}R_p\delta = [\mathfrak{r}_f]_p\} \\
&= -\frac{n^{\frac{k}{2}-1}}{4}\sum_{\pm}\lim_{s\to 0} sm^{1+s}\zeta(1+s)2^{\omega(N)}.
\end{aligned}$$

We can now take the limit without further complications and get the result. $\square$

4.8. The final trace formula.

Without further ado we are now ready to state the final trace formula.

Theorem 4.42 (Eichler's trace formula I). *Let N be square-free, $k > 2$ even, $(n, N) = 1$ and let T_n be the Hecke-operator acting on $S_k(N, Id)$. Then*

$$\begin{aligned}
\mathrm{Tr}(T_n) =& \delta_{n=\square} n^{\frac{k}{2}-1}\frac{k-1}{12}N\psi(N) - \delta_{n=\square}\frac{n^{\frac{k-1}{2}}}{2}2^{\omega(N)} \\
&- \frac{1}{2}\sum_{\substack{t\in\mathbb{Z},\\ t^2-4n<0}}\frac{\lambda_n^+(t)^{k-1}-\lambda_n^-(t)^{k-1}}{\lambda_n^+(t)-\lambda_n^-(t)}\sum_{f^2\mid t^2-4n}\frac{h((t^2-4n)/f^2)}{\sharp(S^\times_{(t^2-4n)/f^2}/\{\pm\})}\prod_{p\mid N}\left(1+\left\{\frac{(t^2-4n)/f^2}{p}\right\}\right) \\
&- \frac{1}{2}\sum_{\substack{t\in\mathbb{Z},\\ t^2-4n=\square}}\frac{\min(|\lambda_n^+(t)|,|\lambda_n^-(t)|)^{k-1}}{|\lambda_n^+(t)-\lambda_n^-(t)|}\sum_{f^2\mid t^2-4n}\phi(f)2^{\omega(N)}.
\end{aligned}$$

Here $\lambda_n^\pm(t)$ are the two (complex) roots of the polynomial $X^2 - tX + n$, $\phi(d) = d\prod_{p\mid d}(1-p^{-1})$ the Euler toitent function, $\psi(d) = \prod_{p\mid d}(1+p^{-1})$ and $\omega(d) = \sharp\{p \mid d\}$. Here $\left\{\frac{\cdot}{p}\right\}$ denotes the modification of the Kronecker-symbol defined in (13).

Proof. This follows directly from the discussion of the foregoing subsections. $\square$

Remark 4.43. Note that a similar formula holds for $k = 2$. However here some subtleties concerning the Bergman kernel arise. In [5, Theorem 2.2] the formula is given as follows:

$$\begin{aligned}\operatorname{Tr}(T_n) =&\delta_{n=\square}\frac{N\psi(N)}{12} - \delta_{n=\square}\frac{n^{\frac{1}{2}}}{2}2^{\omega(N)} + \sigma_1(n)\\ &- \frac{1}{2}\sum_{\substack{t\in\mathbb{Z},\\ t^2-4n<0}}\sum_{f^2|t^2-4n}\frac{h((t^2-4n)/f^2)}{\sharp(S^{\times}_{(t^2-4n)/f^2}/\{\pm\})}\prod_{p|N}\left(1+\left\{\frac{(t^2-4n)/f^2}{p}\right\}\right)\\ &- \frac{1}{2}\sum_{\substack{t\in\mathbb{Z},\\ t^2-4n=\square}}\frac{\min(|\lambda_n^+(t)|,|\lambda_n^-(t)|)}{|\lambda_n^+(t)-\lambda_n^-(t)|}\sum_{f^2|t^2-4n}\phi(f)2^{\omega(N)}.\end{aligned}$$

where as usual N is square-free. This also recovers the dimension formula in [3, (29)]. Note that one can extend the trace formula to $(n,N)\neq 1$, arbitrary N and arbitrary nebentypi. A complete statement is given in [5, Theorem 2.2].

Exercise 7. Use this formula to establish an explicit and a good asymptotic formula for the dimension of $S_k(N,\mathrm{Id})$ when N is square-free and $k>2$ is even.

Proof. The key fact is that $T_1\colon S_k(N,\mathrm{Id})\to S_k(N,\mathrm{Id})$ is the identity. Thus

$$\dim S_k(N,\mathrm{Id}) = \operatorname{Tr}(T_1).$$

This trace can be computed using the trace formula.

Elementary computations show that $t^2-4=\square>0$ has no solutions $t\in\mathbb{Z}$, so that the hyperbolic contribution does not contribute. Furthermore $t^2-4<0$ holds only for $t=0,\pm1$. Thus the elliptic contribution splits essentially in two cases.

First, take $t=0$. In this case we deal with the imaginary quadratic field $K=\mathbb{Q}(i)$ od discriminant -4 and only the maximal order contributes. It is well known that this maximal order $\mathcal{O}_K$ has class number 1 and $\mathcal{O}_K^\times=\{\pm1,\pm i\}$. Further $\lambda_1^{\pm}(0)=\pm i$ are the roots of X^2+1 and one checks that

$$\frac{\lambda_1^+(0)^{k-1}-\lambda_1^-(0)^{k-1}}{\lambda_1^+(0)-\lambda_1^-(0)} = (-1)^{\frac{k}{2}-1}.$$

Second, take $t=\pm1$. In this case we encounter $K=\mathbb{Q}(\sqrt{-3})$ and only the maximal order $\mathcal{O}_K$ contributes. Again we are in a class number 1 situation and the unit group is $\mathcal{O}_K^\times=\{\pm1,\pm\zeta_3,\pm\zeta_3^2\}$. For $\lambda=re^{i\theta}$ we have

$$\frac{\lambda^{k-1}-\overline{\lambda}^{k-1}}{\lambda-\overline{\lambda}} = r^{k-2}\frac{\sin((k-1)\theta)}{\sin(\theta)}.$$

Applying this to the solutions of $X^2+\epsilon+1$ which are $\lambda=\epsilon\zeta_3$, for $\epsilon\in\{\pm1\}$, we find

$$\frac{\lambda^{k-1}-\overline{\lambda}^{k-1}}{\lambda-\overline{\lambda}} = \delta_{3\nmid k-1}(-1)^{\eta+1},$$

where $\eta = k-1 \bmod 3$. In particular the result does not depend on ϵ. This was to expect because we combined the conjugacy classes $t=1$ and $t=-1$ to find a nicer form of the orbital integrals.

Combining everything with the easy scalar and parabolic contribution gives the answer

$$\begin{aligned}\dim S_k(N,\mathrm{Id}) &= \frac{k-1}{12}N\psi(N) - \frac{1}{2}d(N)\\ &\quad + \frac{(-1)^{\frac{k}{2}}}{4}\prod_{p\mid N}\left(1+\left(\frac{-4}{p}\right)\right) + \frac{(-1)^{\eta}}{3}\delta_{3\nmid k-1}\prod_{p\mid N}\left(1+\left(\frac{-3}{p}\right)\right). \qquad (14)\end{aligned}$$

Note that one can check

$$-\frac{(-1)^{\frac{k}{2}}}{4} = \frac{k-1}{4} - \left\lfloor\frac{k}{4}\right\rfloor \text{ and } -\frac{(-1)^{\eta}}{3}\delta_{3\nmid k-1} = \frac{k-1}{3} - \left\lfloor\frac{k}{3}\right\rfloor$$

which makes our formula agree with [3, Chapter III, (29)]. □

Remark 4.44. Another nice application of the trace formula is the vertical Sato-Tate law. This is a result about the distribution of the random variables $S_k(N,\chi) \ni f \mapsto \lambda_f(p)$ as p is fixed and $k+N$ tend to infinity. As long as N is squarefree, $\chi = \mathrm{Id}$ and $k>2$ is even a nice limiting behaviour can ve extracted using the trace formula developed here. Note that one treats all the non-scalar terms essentially trivial for this purpose.

5. Some arithmetic in Quaternion algebras

Our final goal is to define the Brandt matrices and to compute their trace. We will recall the necessary background theory mostly following [9].

5.1. The basics.

Let F be a field with $\mathrm{char}(F) \neq 2$.

Definition 5.1. An algebra B over F is a **quaternion algebra** if it has a basis $\{1_F, i, j, k\}$ over F such that

$$i^2 = a,\ j^2 = b \text{ and } k = ij = -ji,$$

for some $a, b \in F^\times$. If this is the case we write

$$B = \left(\frac{a, b}{F}\right).$$

The primary example to keep in mind are the ordinary quaternions

$$\mathcal{H} = \left(\frac{-1, -1}{\mathbb{R}}\right).$$

Another example is $M_2(F) = \left(\frac{1,1}{F}\right)$ via the isomorphism

$$i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Definition 5.2. An algebra is called **central** if

$$Z(B) = \{z \in B \colon zb = bz \text{ for all } b \in B\} = F.$$

We define the **degree** of an algebra B to be the minimal $m \in \mathbb{N}$ such that every element $b \in B$ satisfies a polynomial $f \in F[X]$ (i.e. $f(b) = 0$) of degree $\leq m$. Further B is called **simple** if it has no non-trivial two-sided ideals (as a ring).

One has the following well known result.

Theorem 5.1. *Let B be an algebra over F. Then the following statements are equivalent:*

- *B is a quaternion algebra;*
- *B is non-commutative and of degree 2;*
- *B is central and of degree 2;*
- *B is central, simple with $\dim_F(B) = 4$.*

Each quaternion algebra B can be equipped with an involution $\alpha \mapsto \overline{\alpha}$ given by

$$\alpha + \beta i + \gamma j + \delta k \mapsto \alpha - \beta i - \gamma j - \delta k.$$

This involution is standard in the sense that $b\overline{b} \in F$ for all $b \in B$. We define

$$\mathrm{nr}(b) = b\overline{b} \text{ and } \mathrm{tr}(b) = b + \overline{b}.$$

Example 5.2. If $B = \left(\frac{a,b}{F}\right)$ one can compute that

$$\mathrm{nr}(\alpha + \beta i + \gamma j + \delta j) = \alpha^2 - a\beta^2 - b\gamma^2 + ab\delta^2 \text{ and } \mathrm{tr}(\alpha + \beta i + \gamma j + \delta j)) = 2\alpha.$$

It is a nice exercise to check some basic properties of this involution and the associated (reduced) norm and (reduced) trace. In particular their relation to $\det(m_b)$ and $\mathrm{Tr}(m_b)$, where $m_b \in \mathrm{End}_F(B)$ is given by $m_b(z) = b \cdot z$.

Remark 5.3. The norm gives rise to a non-degenerate quadratic form of discriminant $1 = a^2b^2 \in F/(F^\times)^2$.

Definition 5.3. We call a quaternion algebra B over F **split** if $B \cong M_2(F)$. A field K is called splitting field for B if $B \otimes_F K \cong M_2(K)$.

Example 5.4. Either $a \in (F^\times)^2$ and $B = \left(\frac{a,b}{F}\right) \cong \left(\frac{1,b}{F}\right) \cong M_2(F)$ is split. Or $a \notin (F^\times)^2$, then $F(\sqrt{a})$ splits B.

Definition 5.4. We define the **Hilbert symbol** $(\cdot,\cdot)_F \colon F^\times \times F^\times \to \{\pm 1\}$ by

$$(a,b)_F = \begin{cases} 1 & \text{if } \left(\frac{a,b}{F}\right) \text{ is split,} \\ -1 & \text{else.} \end{cases}$$

Theorem 5.5. *We have the alternative description of the Hilbert symbol*

$$(a,b)_F = \begin{cases} 1 & \text{if } ax^2 + by^2 = 1 \text{ for some } x, y \in F, \\ -1 & \text{else.} \end{cases}$$

Proof. We omit the proof. However, it is a good exercise to play around with the Hilbert symbol and establish some elementary properties. (For example what is $(a,a)_F$?) ☐

Remark 5.6. The theory for F with even characteristic is similar but slightly more technical. We will not discuss these issues here. Let us only remark, that one should require the generators i, j to satisfy

$$i^2 + i = a,\ j^2 = b \text{ and } ij = j(i+1).$$

Furthermore the Hilbert equation should read $bx^2 + bxy + aby^2 = 1$.

So far we have discussed quaternion algebras over a very general set of fields. Now we will specialise to certain cases important for our goal. These will be $\mathbb{R}$, $\mathbb{Q}_p$ and $\mathbb{Q}$.

The theory over $\mathbb{R}$ should be well known. One of the central results is that the only non-split quaternion algebra over $\mathbb{R}$ is $\mathcal{H}$. We will now prove that the same holds true over $\mathbb{Q}_p$.

We define the p-adic integers by

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \{x = (x_n)_{n\in\mathbb{N}} \colon x_{n+1} \equiv x_n \mod p^n \text{ for all } n \in \mathbb{N}\}.$$

The quotient field is given by $\mathbb{Q}_p = \mathbb{Q}(\mathbb{Z}_p)$. Another more analytic way of thinking about $\mathbb{Q}_p$ is as follows. We equip $\mathbb{Q}$ with the metric $|x|_p = p^{-v_p(x)}$ where $v_p(p^a x') = a$ whenever $(p, x') = 1$ is the p-adic valuation on $\mathbb{Q}$. Then $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \colon |x|_p \leq 1\}$. Obviously $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. Of course v_p extends to a discrete valuation on $\mathbb{Q}_p$ with values in $\mathbb{Z}$. Let K be some (quadratic) extension of $\mathbb{Q}_p$. Then there is a unique valuation w on K which extends v_p. It is given by

$$w(x) = \frac{v_p(\mathrm{Nr}_{K|\mathbb{Q}_p}(x))}{[K \colon \mathbb{Q}_p]}.$$

This fact is usually proven in an algebraic number theory course and we take it for granted.

Lemma 5.7. *Suppose B is a quaternion division algebra over $\mathbb{Q}_p$. Then there is a unique (discrete) valuation $w \colon B \to \mathbb{R} \cup \{\infty\}$ extending v_p. Furthermore, it is given by*

$$w(\alpha) = \frac{v_p(nr(\alpha))}{2}.$$

Proof. We start by showing that w is indeed a discrete valuation. The property that is not completely straight forward is the inequality $w(\alpha+\beta) \geq \min(\alpha, \beta)$. To see this we can assume that $\beta \neq 0$ and compute

$$w(\alpha+\beta) = w(\alpha\beta^{-1}+1) + w(\beta) \geq \min(w(\alpha\beta^{-1}), 0) + w(\beta) = \min(w(\alpha), w(\beta)).$$

Here we used that $K = \mathbb{Q}_p(\alpha\beta^{-1})$ is a quadratic extension of $\mathbb{Q}_p$ and $w|_K$ defines a valuation. Uniqueness can be reduced to to the uniqueness in the quadratic field case by considering $w|_{\mathbb{Q}(\alpha)}$ for $\alpha \in B^\times$. $\square$

Given a quaternion division algebra over $\mathbb{Q}_p$ we define

$$O = \{\alpha \in B \colon w(\alpha) \geq 0\} \text{ and } P = \{\alpha \in B \colon w(\alpha) > 0\}.$$

Then O is a (non-commutative) local ring with unique two sided ideal P. We claim that $P = O\beta$ where β is an element $\beta \in P$ with minimal valuation. Indeed given $\alpha \in P \setminus \{0\}$ we check $w(\alpha\beta^{-1}) = w(\alpha) - w(\beta) \geq 0$. Thus $\alpha\beta^{-1} \in O$ and $\alpha \in O\beta$. Similarly one sees $P = \beta O = O\beta O$.

Theorem 5.8. *There is a unique division algebra B over $\mathbb{Q}_p$. For $p \neq 2$ this is given by*

$$B \cong \left(\frac{e, p}{\mathbb{Q}_p}\right)$$

where e is a quadratic non-residue modulo p.

The proof proceeds by classifying anistropic ternary quadratic forms over $\mathbb{Q}_p$ up to similarity. This can be done in an elementary manner. We will sketch a proof using the theory of valuations.

Proof. By computing the Hilbert symbol we see that $\left(\frac{e,p}{\mathbb{Q}_p}\right)$ is non-split and thus a division algebra. It remains to show uniqueness. Thus, let us take some other quaternion division algebra B' over $\mathbb{Q}_p$. As argued above this comes with a unique valuation w, local ring O' and $P' = \beta O'$. We compute

$$w(\beta) \leq w(p) = v_p(p) = 1 \leq 2w(\beta) = w(\beta^2).$$

This yields the inclusions

$$\beta O' = P' \supset pO' \supset P'^2 = \beta^2 O'.$$

Obviously $O'/P' \cong P'/P'^2$ so that

$$4 = \dim(O'/pO') \leq \dim(O'/P'^2) = 2\dim(O'/P').$$[8]

We conclude that $\dim(O'/P') \geq 2$ with equality if and only if $pO' = P'^2$. We observe that O'/P' is a finite division algebra over $\mathbb{F}_p$ and therefore must be a field (Wedderburn's little theorem). In particular there must be $i \in O'$ such that the reduction $\bar{i} \in O'/P'$ satisfies $O'/P' = \mathbb{F}_p(\bar{i})$. Of course B has degree 2 and i is integral.[9] We conclude that $\mathbb{F}_p(\bar{i})$ is an extension of degree 2 and get $pO' = bloP'^2$, $\dim(O'/P') = 2$ and $w(\beta) = \frac{1}{2}$. Of course $K = \mathbb{Q}_p(i) = \mathbb{Q}_p(\sqrt{e})$ must be the unique unramified quadratic extension of $\mathbb{Q}_p$. By changing variables accordingly we find $B' = \left(\frac{e,b}{\mathbb{Q}_p}\right)$ for some $b \in \mathbb{Z}_p$. Note that (with extra care for $p = 2$) one sees that $v_p(b) \geq 1$ and another suitable change of variables yields the desired equivalence. $\square$

We now turn towards quaternion algebras B over $\mathbb{Q}$. For a prime p we write $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $B_\infty = B \otimes_{\mathbb{Q}} \mathbb{R}$. We say B **ramifies** at $v \in \{p\colon \text{ prime }\} \cup \{\infty\}$ if the completion B_v is a divison algebra. Otherwise, if $B_v \cong M_2(\mathbb{Q}_v)$, we call B **unramified** at v.[10] Let $\mathrm{Ram}(B)$ be the set of places v at which B ramifies.[11] We call B **definite** if $\infty \in \mathrm{Ram}(B)$ and **indefinite** otherwise. We define the **discriminant** H of B by

$$\mathrm{disc}(B) = H = \prod_{p \in \mathrm{Ram}(B) \setminus \{\infty\}} p.$$

From now on let us fix $B = \left(\frac{a,b}{\mathbb{Q}}\right)$. Without loss of generality we can assume $a, b \in \mathbb{Z}$.

Remark 5.9. Note that B is definite if and only if $(a,b)_{\mathbb{R}} = -1$, which happens exactly when $a, b < 0$.

[8]All dimensions are over $\mathbb{F}_p$ and we implicitly use the fact that O' is a $\mathbb{Z}$-lattice of rank 4. This is not hard to see but will also be discussed later on when we are dealing with orders.

[9]Also this fact is borrowed from below.

[10]Note that unramified at v is just a fancy way of saying that B_v is split.

[11]This is a finite set of places, as can be seen using properties of local Hilbert symbols.

Next we derive a remarkable parity restriction for the local Hilbert symbols.

Proposition 5.10 (Hilbert reciprocity)**.** *For all $a, b \in \mathbb{Q}^\times$ we have*

$$(a,b)_{\mathbb{R}} \cdot \prod_p (a,b)_{\mathbb{Q}_p} = 1.$$

We prove this using quadratic reciprocity and a complete understanding of the Hilbert symbol $(\cdot,\cdot)_{\mathbb{Q}_2}$.

Proof. By multiplicativity it is enough to prove this for $a, b \in \{p\colon \text{ prime }\} \cup \{-1\}$. We consider several cases.

First $a = b = -1$. In this case we have

$$(-1,-1)_{\mathbb{Q}_v} = \begin{cases} -1 & \text{if } v = 2, \infty, \\ 1 & \text{else.} \end{cases}$$

Second $a = -1$, $b = p$ this covers also $a = p$, $b = -1$ by symmetry as well as the case $a = b = p$ since $(a,a)_F = (-1,a)_F$. Obviously we find $(-1,p)_{\mathbb{R}} = 1$ and $(-1,p)_{\mathbb{Q}_q} = 1$ for primes $q \neq p$. Furthermore for $q \neq 2$ we have

$$(-1,p)_{\mathbb{Q}_p} = \left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}.$$

At the place 2 we have

$$(-1,p)_{\mathbb{Q}_2} = \begin{cases} 1 & \text{if } p = 2 \text{ or } p \equiv 1 \mod 4 \\ -1 & \text{if } p \equiv 3 \mod 4 \end{cases}$$

Putting everything together concludes this case.

Finally lets look at $a = p$, $q = b$ for (positive) primes $p \neq q$. By direct computation one finds

$$(p,q)_{\mathbb{R}} = 1 \text{ and } (p,q)_{\mathbb{Q}_2} = (-1)^{(p-1)(q-1)/4} = \begin{cases} -1 & \text{if } p, q \equiv 3 \mod 4 \\ 1 & \text{else.} \end{cases}$$

For all primes $l \nmid 2pq$ we have $(p,q)_{\mathbb{Q}_l} = 1$. Thus we are left with

$$(a,b)_{\mathbb{R}} \cdot \prod_p (a,b)_{\mathbb{Q}_p} = (-1)^{(p-1)(q-1)/4} \left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = 1.$$

In the last step we used the quadratic reciprocity law. ☐

Proposition 5.11. *Given a finite set $\Sigma \subset \{p\colon$ prime $\} \cup \{\infty\}$ with an even number of elements. Then there is a quaternion algebra B over $\mathbb{Q}$ with Ram$(B) = \Sigma$.*

Proof. Set

$$D = \prod_{p \in \Sigma \setminus \{\infty\}} p \text{ and } u = \begin{cases} -1 & \text{if } \infty \in \Sigma, \\ 1 & \text{else.} \end{cases}$$

We now choose a prime q such that uq is a quadratic non-residue modulo p or all $2 \neq p \mid D$ and

$$up \equiv \begin{cases} 1 \mod 8 & \text{if } 2 \nmid D, \\ 5 \mod 8 & \text{if } 2 \mid D. \end{cases}$$

Such primes exist due to Dirichlet's theorem concerning primes in arithmetic progressions. Now define

$$B = \left(\frac{uD, uq}{\mathbb{Q}}\right).$$

By construction we have

$$\Sigma \subset \mathrm{Ram}(B) \subset \Sigma \cup \{q\}.$$

According to Hilbert reciprocity $\mathrm{Ram}(B)$ must contain an even number of elements. Thus, since Σ has an even number elements we find $\mathrm{Ram}(B) = \Sigma$ as required. □

Proposition 5.12. *Let B, B' be two quaternion algebras over $\mathbb{Q}$. Then $B \cong B'$ if and only if $B_v \cong B'_v$ for all $v \in \{p\colon \ prime\ \} \cup \{\infty\}$.*

The proof can be reduced to the Hasse-Minkowski theorem, which is a local-to-global principle for quadratic forms. We skip the details.

Combining the last three propositions yields the following nice classification result.

Theorem 5.13. *There is a one-to-one correspondence between Quaternion algebras over $\mathbb{Q}$ up to isomorphism and square-free positive integers H. In other words, a quaternion algebra is uniquely determined by its discriminant H.*

Exercise 8. Consider the quaternion algebra $B = \left(\frac{-2,-37}{\mathbb{Q}}\right)$ and determine the places where it ramifies.

Proof. We first observe that

$$(-2,-37)_{\mathbb{R}} = -1 \text{ and } (-2,-37)_{\mathbb{Q}_{37}} = (-2,-1)_{\mathbb{Q}_{37}} \left(\frac{-2}{37}\right) = -1.$$

In particular we find that

$$\{\infty, 37\} \subset \mathrm{Ram}(B) \subset \{\infty, 2, 37\}.$$

Thus by Hilbert reciprocity we must have $\{\infty, 37\} = \mathrm{Ram}(B)$.[12] In particular, B is definite and has discriminant $H = 37$. □

[12]Of course one can also compute the Hilbert symbol

$$(-2,-37)_{\mathbb{Q}_2} = (-1,-1)_{\mathbb{Q}_2}(2,-1)_{\mathbb{Q}_2}(-1,37)_{\mathbb{Q}_2}(2,37)_{\mathbb{Q}_2} = (-1)\cdot 1 \cdot 1 \cdot (-1) = 1.$$

5.2. **Orders in quaternion algebras.** Let $R \in \{\mathbb{Z}, \mathbb{Z}_p\}$ be a principal ideal domain, take $F = \mathbb{Q}(R)$ and let B be a quaternion algebra over F.

Definition 5.5. A (R-)**lattice** L is a finitely generated submodule $L \subset B$ such that $LF = B$. A lattice $O \subset B$ is called an **order** if it is a subring of B. We call an order **maximal**, if it is not properly contained in any other order.

Example 5.14. We have the following important examples of orders:

(1) Suppose $B \cong M_2(F)$ is split, then $M_2(R)$ is an order in B.
(2) Suppose $B \cong \left(\frac{a,b}{F}\right)$, then $O = R \oplus Ri \oplus Rj \oplus Rk$ is an order in B.
(3) Given a lattice L we define

$$O_l(L) = \{\alpha \in B \colon \alpha L \subset L\} \text{ and } O_r(L) = \{\alpha \in B \colon L\alpha \subset L\}.$$

This is the left order (resp. right order) of L. These orders will be of key importance later on.

Lemma 5.15. *Let $F = \mathbb{Q}$. Then an order $O \subset B$ is maximal if and only if $O_p = O \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is maximal for all primes p.*

Proof. First let O be maximal and suppose that for some prime q we have $O_q \subset O'_q$. We define $O' = O'_q \cap \bigcap_{p \neq q} O_p$. Obviously $O \subset O'$, which implies $O = O'$ and $O_q = O'_q$.

Now we prove the the other direction. Suppose $O \subset O'$. Then we have $O_p \subset O'_p$ for all p and since O_p is maximal we must have $O_p = O'_p$. We conclude that

$$O = \bigcap_p O_p = \bigcap_p O'_p = O'.$$

□

Lemma 5.16. *Let $F = \mathbb{Q}$ and $O \subset B$ be an order. Then there is a maximal order $O' \subset B$ containing O. In particular, maximal orders exist. Furthermore, O_p is maximal for all but finitely many p.*

Proof. We leave the proof as an exercise for the reader. □

Thus we need to study orders in quaternion algebras in non-archimedean fields $F = \mathbb{Q}_p$. Let us start with the split case $B = M_2(\mathbb{Q}_p)$. In this case we have a very general construction of orders. Indeed let V be a F-vector space with $\dim_F(V) = 2$. Then $B \cong \mathrm{End}_F(V)$. Given a $\mathbb{Z}_p$-lattice $L \subset V$ we define

$$\mathrm{End}_R(L) = \{f \in \mathrm{End}_F(V) \colon f(L) \subset L\}.$$

It can be shown, that this is an order.

Lemma 5.17. *Let $B = End_F(V)$ as above. Every maximal order O in B is of the form $O = End_R(L)$ for some lattice $L \subset V$. Furthermore, every maximal order in $M_2(\mathbb{Q}_p)$ is conjugate to $M_2(\mathbb{Z}_p)$.*

Proof. We show that every order is contained in $\mathrm{End}_F(L)$ for some lattice $L \subset V$. The first statement then follows from maximality while the second one can be obtained by a suitable change of basis.

Given any lattice N and any order O' we define the lattice $L = \{x \in N : O'x \subset N\}$. By definition we have $O' \subset \mathrm{End}_F(L)$. □

Example 5.18. Another important example for an order in $M_2(\mathbb{Q}_p)$ is given by

$$\mathfrak{M}_0(p^k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : c \in p^k\mathbb{Z}_p \right\}.$$

Note that this order can be constructed as the endomorphism ring of a lattice.

We not turn to the non-split situation. We are still considering $F = \mathbb{Q}_p$ but now we are assuming that B is a division algebra. Here we have the following important result.

Lemma 5.19. *Let $O = \{\alpha \in B : \alpha$ is integral over $\mathbb{Z}_p\}$. Here an element $\alpha \in B$ is called integral if there is a monic $P \in \mathbb{Z}_p[X]$ such that $P(\alpha) = 0$. Then O is the unique maximal order in B.*

Proof. It can be seen that all elements of an order must be integral. Thus, if we succeed to show that O is an order it must be the unique maximal order, since it contains all orders. We claim that O coincides with the valuation ring of B. If this is shown O is obviously a ring and satisfies $\mathbb{Q}_pO = B$. Furthermore, since all elements of O are integral it follows that it must be an order.[13]

Thus we need to show that $\alpha \in O$ if and only if $w(\alpha) \geq 0$. To show this we first suppose that α is integral. Then the minimal polynomial $f_\alpha(X) = X^2 - \mathrm{tr}(\alpha)X + \mathrm{nr}(\alpha) \in \mathbb{Z}_p[X]$. In particular $\mathrm{nr}(\alpha) \in \mathbb{Z}_p$ and $w(x) = \frac{\mathrm{nr}(\alpha)}{2} \geq 0$. On the other hand, if $w(\alpha) \geq 0$ we set $K = \mathbb{Q}_p(\alpha)$ and obviously $0 \leq w(\alpha) = w|_K(\alpha)$. Since the ring of integers of K coincides with the valuation ring $w|_K^{-1}(\mathbb{R}_{\geq 0})$ we are done. □

Corollary 5.20. *We can write $O = S_K + S_Kj$, where S_K is the ring of integers in $K = \mathbb{Q}_p(\sqrt{e})$ the unique unramified extension of $\mathbb{Q}_p$ and $P = Oj$ is the unique maximal ideal of O.*

Proof. By Theorem 5.8 we have $B \cong \left(\frac{e,p}{\mathbb{Q}_p}\right)$. We can rewrite this as $B = K + Kj$ with $j^2 = p$. In particular, given $\alpha = u + vj$ we have $\mathrm{nr}(\alpha) = \mathrm{nr}(u) - p\mathrm{nr}(v)$. Now the p-adic valuation of $\mathrm{nr}(u)$ is even while the one of $p\mathrm{nr}(v)$ is odd. We conclude that $v_p(\mathrm{nr}(\alpha)) \geq 0$ if and only if $v_p(\mathrm{nr}(v)) \geq 0$ and $v_p(\mathrm{nr}(u)) \geq 0$. We are done since $S_K = \{x \in K : v_p(\mathrm{nr}(x)) \geq 0\}$. □

Example 5.21. Let $B = \left(\frac{-1,-1}{\mathbb{Q}_2}\right)$. One can show that the element $w = \frac{1}{2}(-1 + i + j + k)$ satisfies the $\mathbb{Z}_2$-integral equation $w^2 + w + 1 = 0$. One concludes that

[13] It makes a nice exercise in algebra to show that a ring O consisting of integral elements with $\mathbb{Q}_pO = B$ is finitely generated.

the order $\mathbb{Z}_2 + \mathbb{Z}_2 i + \mathbb{Z}_2 j + \mathbb{Z}_2 k$ is not maximal. On the other hand the unique maximal order is given by $O = \mathbb{Z}_2 + \mathbb{Z}_2 i + \mathbb{Z}_2 j + \mathbb{Z}_2 w$.

We now turn to the global situation. Let $F = \mathbb{Q}$ and B be a quaternion algebra over $\mathbb{Q}$. Let us start with a simple example.

Example 5.22. The lattice $L = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ is an order, the so called Lipschitz order, in B, but is not maximal. It is contained in the maximal order

$$O = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}w,$$

for $w = \frac{-1+i+j+k}{2}$. This order is called the Hurwitz order.

In general orders in B are very wild objects. Therefore we restrict ourselves to special classes of orders.

Definition 5.6. An order $O \subset B$ is called **hereditary** if for all p we have O_p is maximal or $O_p \cong \mathfrak{M}_0(p) \subset M_2(\mathbb{Q}_p) \cong B_p$. More generally we call O an **Eichler order** if O_p is maximal or $O_p \cong \mathfrak{M}_0(p^k) \subset M_2(\mathbb{Q}_p) \cong B_p$.

Definition 5.7. Let $O \subset B$ be an order with $\mathbb{Z}$-basis $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Then we define

$$\mathrm{disc}(O) = |\det(\mathrm{tr}(\alpha_i \alpha_j))_{i,j}| \in \mathbb{N}.$$

Note that in general the discriminant is an ideal instead of a ring element. However, since all the rings we are considering are principal ideal domains we can make this simplification here.

Example 5.23. If $B = \left(\frac{a,b}{\mathbb{Q}}\right)$ and $O = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$, then

$$\mathrm{disc}(O) = (4ab)^2.$$

Proposition 5.24. *An order O is maximal if and only if* $\mathrm{disc}(O) = \mathrm{disc}(B)^2$. *Furthermore, O is hereditary if and only if* $\mathrm{disc}(O) = M^2 \,\mathrm{disc}(B)^2$ *with M square-free and* $(\mathrm{disc}(B), M) = 1$.

The proof reduces to local computations of discriminants using the following two lemmata. We leave the details to the reader.

Lemma 5.25. *Let $O \subset O'$ be two orders in B. Then*

$$\mathrm{disc}(O) = [O' : O]_{\mathbb{Z}}^2 \,\mathrm{disc}(O'),$$

for $[O' : O]_{\mathbb{Z}} = \sharp(O'/O)$.[14] *Furthermore, $O = O'$ if and only if* $\mathrm{disc}(O) = \mathrm{disc}(O')$.

Lemma 5.26. *We have* $\mathrm{disc}(O_p) = p^{v_p(\mathrm{disc}(O))}$ *and* $\mathrm{disc}(O) = \prod_p \mathrm{disc}(O_p)$.

[14]From this one can derive that $\mathrm{disc}(O)$ is always a square. Thus some authors prefer to define the reduced discriminant which essentially is $\mathrm{discrd}(O) = \sqrt{\mathrm{disc}(O)}$. Of course there is also a proper algebraic construction of the reduced discriminant.

We end this subsection on orders by quickly discussing the unit group $O^\times$ of an order $O \subset B$, for definite B. Recall that we have the inclusion $B \subset B_\infty$. Since B is definite B_∞ is a divison algebra, so that $\mathrm{nr}\colon B \to \mathbb{R}$ defines an anisotropic quadratic form. Since B_∞ is a finite dimensional real vector space nr must be definite. Further $\mathrm{nr}(1) = 1$, which implies that nr is positive definite.

Now suppose $u \in O^\times$. Then $\mathrm{nr}(u), \mathrm{nr}(u^{-1}) \in \mathbb{Z}_+$, since O is an order. Further by multiplicativity of the norm we must have $\mathrm{nr}(u) \cdot \mathrm{nr}(u^{-1}) = 1$. Thus $\mathrm{nr}(u) = 1$. Since $\mathcal{O}$ is a lattice and $\{x \in B_\infty \colon \mathrm{nr}(x) = 1\}$ is compact we find that $O^\times$ is finite.

Note that more can be said. Indeed, by uniqueness we know that $B_\infty \cong \mathcal{H}$. But the unit group of the Hamiltonian quaternions and its finite subgroups are well understood. One can prove the following theorem.

Theorem 5.27. *Let O be a $\mathbb{Z}$-order in a definite quaternion algebra B. Then $O^\times/\{\pm 1\}$ is one of the following:*

- *Cyclic of order* 2,4 *or* 6*;*
- *Quaternion of order* 8*;*
- *Binary dihedral of order 12;*
- *Binary tetrahedral of order 24.*

Exercise 9. Let $B = \left(\frac{-2,-37}{\mathbb{Q}}\right)$. Show that

$$O = \frac{1}{2}(1 + j + k)\mathbb{Z} + \frac{1}{4}(i + 2j + k)\mathbb{Z} + j\mathbb{Z} + k\mathbb{Z} \tag{15}$$

is a maximal order.

Proof. We first compute the matrix

$$A = (\mathrm{tr}(\alpha_i\alpha_j))_{i,j} = -\begin{pmatrix} 54 & 37 & 37 & 74 \\ 37 & 28 & 37 & 37 \\ 37 & 37 & 74 & 0 \\ 74 & 37 & 0 & 148 \end{pmatrix}.$$

Now $\mathrm{disc}(O) = |\det(A)| = 1396 = 37^2$. Since $\mathrm{disc}(B) = 37$ we find that O is maximal. $\square$

5.3. **Ideals in quaternion orders.** Let B be a finite dimensional $\mathbb{Q}$-algebra. It would make sense to consider more generally finite dimensional F-algebras, for a number field F, but for our purposes $F = \mathbb{Q}$ suffices.

We will now have to talk about ideals and ideal classes in orders. The goal is to somehow generalise the theory known for number fields. It is however necessary to properly deal with the non-commutativity.

Definition 5.8. We call a lattice L **principal** if there is $\alpha \in B$ such that

$$L = O_l(L)\alpha = \alpha O_r(L).$$

We call α the **generator** of L. We call L **locally principal** if L_p is principal for all p. We call L **integral** if $L \subset O_l(L) \cap O_r(L)$.

Remark 5.28. It is a nice exercise to observe that the obvious notions of left-integral and left-principal are equivalent to our definition. Further one can observe that L is integral if and only if it is a right (resp. left) $O_r(L)$ ideal (resp. $O_l(L)$ ideal).

Definition 5.9. let O be an order in B. A **left-fractional-O-ideal** (resp. **right-fractional-O-ideal**) is a lattice $L \subset B$ such that $O \subset O_l(L)$ (resp. $O \subset O_r(L)$). Given two order O, O' we say L is a **fractional-O, O'-ideal** if it is a left-fractional-O-ideal and a right-fractional-O'-ideal. We call L **sated** (as a left-fractional-O-ideal) if $O = O_l(L)$. We can extend the notion of satedness in the obvious way to right-fractional-O-ideals and fractional-O, O'-ideals.

Definition 5.10. Let $L, J \subset B$ be two lattices. We say L is **compatible** with J if $O_r(L) = O_l(J)$. We set

$$LJ = \{\alpha\beta \colon \alpha \in L, \beta \in J\}.$$

Definition 5.11. We call a lattice $L \subset B$ **right** (resp. left) **invertible** if there is a lattice $L' \subset B$ such that L is compatible with L' (resp. L' is compatible with L) and $LL' = O_l(L)$ (resp. $L'L = O_r(L)$). We call L' the right (resp. left) inverse of L. We say L is invertible if there is a lattice L' which is simultaneously right and left inverse for L. We call L' the two-sided inverse and write $L^{-1} = L'$. It is given by

$$L^{-1} = \{\alpha \in B \colon L\alpha L \subset L\}.$$

Remark 5.29. It is a nice exercise to show that principal lattices are invertible.

Definition 5.12. A left (resp. right) fractional O-ideal is lattice $L \subset B$ with $O \subset O_l(L)$ (resp. $O \subset O_r(L)$). We define

$$\mathrm{nr}(L) = \gcd(\{\mathrm{nr}(\alpha) \colon \alpha \in L\}).$$

For an integral ideal L we define the **absolute norm** by

$$N(L) = \sharp(O_l(L)/L).$$

If L is fractional there is $\alpha \in B^\times$ with αL integral and we define in an ad-hoc manner[15]

$$N(L) = \frac{N(\alpha L)}{\mathrm{nr}(\alpha)^2}.$$

Remark 5.30. If L is principal with generator α, then we have $\mathrm{nr}(L) = \mathrm{nr}(\alpha)$. More generally one has $\mathrm{nr}(\alpha L) = \mathrm{nr}(\alpha)\mathrm{nr}(L)$. Further $\mathrm{nr}(L_1L_2) \mid \mathrm{nr}(L_1)\mathrm{nr}(L_2)$ but equality does not hold in general. To prove these facts is a nice exercise.

Lemma 5.31. *Suppose L_1 is compatible with L_2 and L_1 or L_2 is locally principal, then $nr(L_1L_2) = nr(L_1)nr(L_2)$.*

[15]One can correctly define the $\mathbb{Z}$-module $\mathrm{Nm}_{B/F}(L) = \{\mathrm{nr}(\alpha)^2 \colon \alpha \in L\}$ the absolute norm is then given by $N(\mathrm{Nm}_{B/F}(L))$. This works also over number fields.

Proof. Note that the equality $\mathrm{nr}(\alpha L) = \mathrm{nr}(\alpha)\mathrm{nr}(L)$ holds also locally. Thus one completes the proof by observing that $\mathrm{nr}(L) = \prod_p \mathrm{nr}(L_p)$. $\square$

In the split case we have the following nice result concerning principal lattices.

Proposition 5.32. *Let R be a principal ideal domain and $F = \mathbb{Q}(R)$ be its field of fractions. Further let $L \subset M_2(F)$ be an R-lattice such that $O_l(L)$ or $O_r(L)$ is maximal. Then L is principal and both $O_l(L)$ and $O_r(L)$ are maximal.*

Proof. Without loss of generality we assume that L is integral and $O_l(L) = M_2(R)$ is maximal (rescaling and conjugating).

We choose a set of generators $\alpha_1, \ldots, \alpha_m$ of L and define the matrix

$$A = (\alpha_1 \cdots \alpha_m)^t \in M_{2m\times 2}(R).$$

Bringing this matrix in Hermite normal form yields $Q \in \mathrm{GL}_{2m}(R)$ with $QA = (\beta, 0)^t$ for $\beta \in M_2(R)$. One concludes the proof by showing

$$L = M_2(R)\beta.$$

We leave the details as an exercise. $\square$

In a similar direction one has the following important result.

Proposition 5.33. *A lattice L is invertible if and only if L is locally principal.*

Corollary 5.34. *It L_1 or L_2 is invertible, then $nr(L_1L_2) = nr(L_1)nr(L_2)$.*

Definition 5.13. We call a fractional-O, O'-ideal L **invertible** if it is invertible as a lattice and sated.

Remark 5.35. It can be shown, that for a maximal order O all left (or right) fractional O-ideals are invertible. This does not remain true in general.

Definition 5.14. Two lattices $L_1, L_2 \subset B$ are in the same **right class**, if $\alpha L_1 = L_2$ for some $\alpha \in B^\times$. We write $L_1 \sim_r L_2$. (Similarly one defines left classes.) This defines an equivalence relation and we denote its classes by $[\cdot]_r$. We define the **right class set** of O by

$$\mathrm{Cls}_r(O) = \{[L]_r \colon L \text{ is an invertible right-fractional-}O\text{-ideal }\}.$$

Our task is now to show that the right class set is finite. To do so we will employ the geometry of numbers. Let us start by giving a crash course in it.

A lattice $\Lambda \subset \mathbb{R}^n$ is a discrete subgroup such that $\mathbb{R}^n/\Lambda$ is compact. Alternatively we can say $\Lambda \cong \mathbb{Z}^n$ and $\mathbb{R}\Lambda = \mathbb{R}^n$. We define

$$\mathrm{covol}(\Lambda) = \mathrm{vol}(\mathbb{R}^n/\Lambda).$$

More concretely there is a basis $\alpha_1, \ldots, \alpha_n$ of $\mathbb{R}^n$ such that $\Lambda = \bigoplus_i \mathbb{Z}\alpha_i$ and $\mathrm{covol}(\Lambda) = |\det(\alpha_{ij})_{i,j}|$.

We will use the following theorem.

Theorem 5.36 (Minkowski)**.** *Let $X \subset \mathbb{R}^n$ be a closed convex symmetric subset and let $\Lambda \subset \mathbb{R}^n$ be a lattice. If $vol(X) \geq 2^n covol(\Lambda)$ then there is $0 \neq \alpha \in \Lambda \cap X$.*

Proposition 5.37. *Let B be a definite Quaternion algebra over $\mathbb{Q}$ with an order $O \subset B$. Then every ideal class in $Cls_r(O)$ is represented by an integral right-O-ideal L with*

$$N(L) \leq \frac{8}{\pi^2}\sqrt{disc(O)}.$$

In particular, the right class set $Cls_r(O)$ is finite.

Proof. Let $B = \left(\frac{a,b}{\mathbb{Q}}\right)$. Since B is supposed to be definite we can assume $a, b \in \mathbb{Z}_{<0}$. We have

$$B_\infty = B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathcal{H}.$$

More concretely we can embed $B_\infty \hookrightarrow \mathbb{R}^4$ via

$$t + xi + yj + zk \mapsto \sqrt{2}(t, x\sqrt{-a}, y\sqrt{-b}, z\sqrt{ab}).$$

It is easy to verify that $2\mathrm{nr}(\alpha) = \|\alpha\|^2$, where $\|\cdot\|$ is the usual euclidean norm on $\mathbb{R}^4$. Further we compute

$$\mathrm{disc}(O) = \mathrm{covol}(O)^2,$$

where we view O as a lattice in $\mathbb{R}^4$ under the embedding above. This is obvious if we can choose a diagonal basis for O.

Let J be an invertible right-fractional-O-ideal. We need to find an integral ideal L with small absolute norm in the same class of J. To do so we set $c^4 = \frac{32}{\pi^2}\mathrm{covol}(J^{-1})$ and $X = B_c(0)$. This is obviously closed, symmetric and convex with

$$\mathrm{vol}(X) = 16\mathrm{covol}(J^{-1}),$$

so that Minkowski's theorem tells us that there is $\alpha \in (J^{-1} \cap X) \setminus \{0\}$. We put $L = \alpha J$. By construction we have $[L]_r = [J]_r$ and L is obviously integral. We have to estimate the absolute norm:

$$N(\alpha J) = \mathrm{nr}(\alpha)^2 N(J) = \frac{1}{4}\|\alpha\|^4 N(J) = \frac{1}{4}\|\alpha\|^4 \frac{\mathrm{disc}(O)^{\frac{1}{2}}}{\mathrm{covol}(J^{-1})} \leq \frac{8}{\pi^2}\mathrm{disc}(O)^{\frac{1}{2}}.$$

The finiteness of the right class set follows since there are only finitely many integral-right-O-ideals of given norm. ☐

Definition 5.15. We say two orders are of the same **type** if there is $\alpha \in B^\times$ such that $O' = \alpha^{-1}O\alpha$. We write $O \equiv O'$. We call O and O' **connected** if $O'_p \equiv O_p$ for all p. We set

$$\mathrm{Gen}(O) = \{O' \colon O' \text{ is connected to } O\}.$$

We define the **type-set** of O, denoted by $\mathrm{Typ}(O)$, to be a set of different types of orders making up the genus of O.[16]

[16] Of course being of the same type defines an equivalence relation on the set of all orders and thus in particular on $\mathrm{Gen}(O)$. The type set is exactly given by $\mathrm{Gen}(O)$ modulo this equivalence relation.

Example 5.38. There is a unique genus consisting exactly of maximal orders.

Lemma 5.39. *Let $O' \in Gen(O)$ then $\sharp Cls_r(O) = \sharp Cls_r(O')$.*

Proof. The key is to construct a locally principal fractional O,O' ideal $J \subset B$. Once one found such an ideal, also called a connecting ideal, one obtains the bijection

$$\mathrm{Cls}_r(O) \to \mathrm{Cls}_r(O'),\ [I]_r \mapsto [IJ]_r.$$

It is a nice exercise to construct J using the assumption that O and O' are connected. One can even go further and show that O and O' are connected if and only if such an ideal J exists. □

Remark 5.40. One can show that the map

$$\mathrm{Cls}_r(O) \to \mathrm{Typ}(O), [I]_r \mapsto \text{ class of } O_l(I)$$

is surjective. In particular, the set $\mathrm{Typ}(O)$ is finite.

We end this section by giving an idelic incarnation of the right-class-set. To do so we quickly introduce some notation. We write

$$\mathbb{A}_{B,f}^\times = {\prod_p}' B_p^\times = \{(b_p)_p\colon b_p \in O_p^\times \text{ for all but finitely many } p\},$$
$$\mathbb{A}_B^\times = B_\infty \times \mathbb{A}_{B,f}^\times \text{ and } \widehat{O}^\times = \prod_p O_p^\times.$$

These are the **finite ideles** and the **ideles** of B. Note that $B^\times$ embeds diagonally in $\mathbb{A}_B^\times$ and $\mathbb{A}_{B,f}^\times$. Further $O^\times$ embeds diagonally in $\widehat{O}^\times$. Suppressing these embeddings from the notation we have

$$B^\times \cap \widehat{O}^\times = O^\times.$$

Lemma 5.41. *There is a bijection*

$$Cls_r(O) \leftrightarrow B^\times \backslash \mathbb{A}_{B,f}^\times / \widehat{O}^\times.$$

Proof. Take an invertible right-fractional-O-ideal I. Then I is locally principal so that we can write $I_p = \alpha_p O_p$ for $\alpha_p \in B_p^\times$, which are well defined up to right multiplication by $O_p^\times$. Thus we get a well defined map

$$\{ \text{ invertible right-fractional-}O\text{-ideals } \} \to \mathbb{A}_{B,f}^\times / \widehat{O}^\times,\ I \mapsto (\alpha_p)_p.$$

This map descents to the desired bijection. □

This is completely analogue to the situation in quadratic fields K (or more generally separable Q algebras of dimension 2) with an $(\mathbb{Z})$-order $\mathfrak{r}$. Indeed here one has an group-isomorphism

$$\mathrm{Cls}(\mathfrak{r}) \cong K^\times \backslash \mathbb{A}_{K,f}^\times / \widehat{\mathfrak{r}}^\times. \tag{16}$$

Here the definitions are completely analogous.

$$\mathbb{A}_{K,f}^{\times} = \prod_p{}' [K \otimes \mathbb{Q}_p], \text{ and } \widehat{\mathfrak{r}}^{\times} = \prod_p [\mathfrak{r} \otimes \mathbb{Z}_p]^{\times}.$$

The adeles and ideles turn out particular useful if one wants to work over more general number fields than $\mathbb{Q}$. However, they form a flexible framework to efficiently implement the local-to-global arguments that we have already used so many times.

We conclude this section by proving an adelic compactness result which also implies the finiteness of the class number. Note that the ideles come with a multiplicative map

$$\|\cdot\| \colon \mathbb{A}_B^{\times} \to \mathbb{R}_{>0},\ (\alpha_v)_v \mapsto \prod_v |\mathrm{nr}(\alpha_v)|_v.$$

We define $\mathbb{A}_B^{(1)} = \ker(\|\cdot\|)$. Of course we have the inclusion $B^{\times} \subset \mathbb{A}_B^{(1)}$ diagonally.

Theorem 5.42. *Let B be a division algebra over $\mathbb{Q}$, then $B^{\times} \subset \mathbb{A}_B^{(1)}$ is co-compact.*

Remark 5.43. One deduces easily that $B^{\times} \subset \mathbb{A}_{B,f}^{\times}$ is co-compact which on the other hand implies that $B^{\times} \backslash \mathbb{A}_{B,f}^{\times} / \widehat{O}^{\times}$ is finite. This is a standard topological argument!

Proof. In this proof we will use the following existence statement which can be interpreted as an adelic version of Minkowski's convex body theorem. There is a compact set $E \subset \mathbb{A}_B$ such that for all $\beta \in \mathbb{A}_B^{(1)}$ the map $\beta E \hookrightarrow \mathbb{A}_B \to B\backslash\mathbb{A}_B$ is not injective.

Let E be as above and set $X = E - E$. Note that X as well as $X \cdot X$ are compact in $\mathbb{A}_B$.

By construction of E there exist distinct elements $x, x' \in E$ such that $\beta(x-x') \in B \setminus \{0\} = B^{\times}$. Thus, for all $\beta \in \mathbb{A}_B^{(1)}$ we know that $\beta X \cap B^{\times} \neq \emptyset$.

Define $T = B^{\times} \cap X \cdot X$. Since $X \cdot X$ is compact and $B^{\times}$ is discrete, T must be a finite set. Further we put

$$K = T^{-1} \cdot X \times X,$$

which is obviously finite. Now given $\beta \in \mathbb{A}_B^{(1)}$ we know that $\beta X \cap B^{\times} \neq \emptyset$ and similarly $X\beta^{-1} \cap B^{\times} \neq \emptyset$. Unravelling this we find $v, v' \in X$ and $b, b' \in B^{\times}$ such that $\beta v = b$ and $v'\beta^{-1} = b'$. In particular

$$b'b = (v'\beta^{-1}) \cdot (\beta v) = v'v \in B^{\times} \cap X \cdot X = T.$$

This implies $v^{-1} \in T^{-1}X$ and $\beta = bv^{-1}$. Thus we have seen that we can decompose $\mathbb{A}_B^{(1)} = B^{\times} d^{-1}(K)$ where $d \colon \mathbb{A}_B^{(1)} \ni x \mapsto (x, x^{-1})\mathbb{A}_B^{\times} \times \mathbb{A}_B^{\times}$.

In particular we have a surjection $d^{-1}(K) \to B^{\times}\backslash\mathbb{A}_B^{(1)}$. Since the set $d^{-1}(K)$ is compact we are done. $\square$

5.4. **Eichler's mass formula.** We are now going to prove a formula for the (weighted) number of (right)-classes in an order O. This can be seen as a quaternionic analogue of Dirichlet's class number formula for imaginary quadratic fields.

Definition 5.16. Given an order O we define the **zeta-function of** O by

$$\zeta_O(s) = \sum_{I \subset O} \frac{1}{N(I)^s} = \sum_{n=1}^{\infty} \frac{a_n(O)}{n^{2s}}$$

with

$$a_n(O) = \sharp\{I \subset O \colon \mathrm{nr}(I) = n\}.$$

If O is a maximal order this corresponds to the Dedekind zeta function of an imaginary quadratic field.

Lemma 5.44. *If $O' \in Gen(O)$, then $a_n(O) = a_n(O')$ for all $n \in \mathbb{N}$. In particular, ζ_O depends only on the genus of O.*

Proof. Using the local to global properties of lattices it is easy to construct a bijection between $\{I \subset O \colon \mathrm{nr}(I) = n\}$ and $\{I \subset O' \colon \mathrm{nr}(I) = n\}$. ☐

Lemma 5.45. *Let I be an invertible integral lattice with $nr(I) = nm$ for integer $(n, m) = 1$. Then there is a unique integral lattice J such that I is compatible with J^{-1} and IJ^{-1} is integral and $nr(J) = m$.*

Proof. The lattice J is constructed locally. ☐

Corollary 5.46. *For $(n, m) = 1$ we have $a_{nm}(O) = a_n(O)a_m(O)$.*

Proof. Let $A_n(O) = \{I \subset O \colon \mathrm{nr}(I) = n\}$. We have a map

$$A_{nm}(O) \to A_n(O),\ I \mapsto J,$$

where J is the unique lattice constructed in the previous lemma. One can show that each fibre of this map has cardinality $a_m(O)$. ☐

This obviously implies that $\zeta_O(s)$ is eulerian. Thus we can write

$$\zeta_O(s) = \prod_p \zeta_{O_p}(s).$$

If O is a maximal order the numbers $a_n(0)$ can be computed explicitly and one finds

$$\zeta_O(s) = \zeta(2s)\zeta(2s-1) \prod_{p \mid \mathrm{disc}(B)} (1 - p^{1-2s}).$$

On the way one establishes that, for $(n, \mathrm{disc}(O)) = 1$, we have

$$a_n(O) = \sigma_1(n).$$

Lemma 5.47. *For a maximal order O we have*

$$\zeta_O^*(1) = \lim_{s\downarrow 0}(s-1)\zeta_O(s) = \pi^2\frac{\varphi(H)}{12H},$$

where $H = \mathrm{disc}(B)$.

Proof. From the explicit formula above we see that

$$\zeta_O^*(1) = \underbrace{\zeta(2)}_{=\frac{\pi^2}{6}}\frac{\varphi(H)}{H}\underbrace{\lim_{s\downarrow 0}(s-1)\zeta(2s-1)}_{=\frac{1}{2}\zeta^*(1)=\frac{1}{2}}.$$

□

The main reason behind defining these quaternionic zeta functions is to generalise Dirichlet's class number formula for imaginary quadratic fields. Indeed one can prove the following weighted class number formula for Eichler orders.

Theorem 5.48. *Let B be a definite Quaternion algebra over $\mathbb{Q}$ and let O be an Eichler order with* $\mathrm{disc}(O) = M^2D^2$, *where* $D = \mathrm{disc}(B)$ *and* $(M, D) = 1$. *Then*

$$\mathit{mass}(\mathit{Cls}_r(O)) = \sum_{[J]_r\in \mathit{Cls}_r(O)}\frac{1}{w_J} = \frac{\varphi(D)M\psi(M)}{12},$$

where $w_J = \sharp(O_l(J)^\times/\{\pm 1\})$ *and φ is the Euler toitent function and* $\psi(M) = \prod_{p\mid M}(1+\frac{1}{p})$.

Before we can prove this theorem we need some preparation.

Definition 5.17. We define the **partial zeta-function of** O as follows. Let J be an integral invertible right-O-ideal. Then we define

$$\zeta_{O,[J]_r}(s) = \sum_{\substack{I\subset O,\\ [I]_r=[J]_r}} N(I)^{-s}.$$

Remark 5.49. These new functions are built such that

$$\zeta_O(s) = \sum_{[J]_r\in \mathrm{Cls}_r(O)}\zeta_{O,[J]_r}(s).$$

On the other hand, since $\mu J = J$ if and only if $\mu \in O_l(J)^\times$ and $[I]_r = [J]_r$ exactly if $I = \alpha J$ for $\alpha \in J^{-1}$, we have

$$\zeta_{O,[J]_r}(s) = \frac{1}{w_J N(J)^s}\sum_{0\neq\alpha\in J^{-1}/\{\pm 1\}}\mathrm{nr}(\alpha)^{-2s}.$$

We will need to understand the analytic properties of these functions. We will do this by invoking the following theorem, which we don't prove here.

Theorem 5.50. *Let Λ be a $\mathbb{Z}$-lattice, $X \subset \mathbb{R}^n$ be a cone and $N\colon X \to \mathbb{R}_{>0}$ be a function such that $N(tx) = t^n N(x)$ for all $x \in X$ and $t \in \mathbb{R}_+$. We set $X_{\leq 1} = \{x \in X\colon N(x) \leq 1\}$. Then*

$$\zeta_{\Lambda,X}(s) = \sum_{X\cap\Lambda} \frac{1}{N(\Lambda)^s}$$

converges for $\mathrm{Re}(s) > 1$ *and*

$$\zeta^*_{\Lambda,X}(1) = \frac{vol(X_{\leq 1})}{covol(\Lambda)}.$$

Lemma 5.51. *We have*

$$\zeta^*_{O,[J]_r}(1) = \frac{\pi^2}{w_J \operatorname{disc}(O)^{\frac{1}{2}}}.$$

Proof. First we can identify J^{-1} with a lattice $\Lambda \subset \mathbb{R}^4$. This is done as in the proof of Proposition 5.37. There we did also see that

$$\operatorname{covol}(\Lambda) = \frac{\operatorname{disc}(O)^{\frac{1}{2}}}{N(J)}.$$

Recall that under this identification we also have $2\operatorname{nr}(\alpha) = \|\alpha\|^2$. We now take the cone X by suitably choosing a fundamental domain for the action of $\{\pm 1\}$ on $B_{\mathbb{R}} \cong \mathbb{R}^4$. Putting $N = \|\cdot\|^4$, we find

$$\zeta_{O,[J]_r}(s) = \frac{1}{w_J(4N(J))^s}\zeta_{\Lambda,X}(s).$$

The above theorem implies the statement since $\mathrm{vol}(X_{\leq 1}) = \frac{\pi^2}{4}$. □

Proposition 5.52. *If O is a maximal order, then*

$$mass(Cls_r(O)) = \frac{\varphi(H)}{12},$$

where H is the discriminant of B.

Proof. Since $\operatorname{disc}(O) = H^2$ for maximal orders we have

$$\frac{\varphi(H)}{12} = \frac{H}{\pi^2}\zeta^*_O(1) = \frac{H}{\pi^2}\sum_{[J]_r\in \mathrm{Cls}_r(O)} \zeta^*_{O,[J]_r}(1) = \mathrm{mass}(\mathrm{Cls}_r(O)).$$

□

We will now reduce the general case to the one of maximal orders as follows.

Definition 5.18. We call $O' \supset O$ a **($\mathbb{Z}$)-superorder** if there is a prime l such that $O'_p = O_p$ for all $p \neq l$.

Lemma 5.53. *If O' is a superorder of O, then we have*

$$mass(Cls_r(O)) = [(O'_l)^\times : O_l^\times]\, mass(Cls_r(O')).$$

Proof. We have the surjective map

$$\mathrm{Cls}_r(O) \to \mathrm{Cls}_r(O'),\ [I]_r \mapsto [IO']_r.$$

We need to understand the fibres of this map. Without loss of generality we can work with suitable representatives. The fibres of this map (on the level of representatives) are

$$F(I') = \{I \subset O \colon IO' = I'\}.$$

Given $\mu_l \in (O'_l)^\times$ we define

$$[I^{\mu_l}]_p = \begin{cases} \beta_l \mu_l O_l & \text{if } p = l \text{ and } I_p = \beta_p O_p, \\ I_p & \text{if } p \neq l. \end{cases}$$

This defines a simply transitive right action of $(O'_l)^\times$ on $F(I')$. The kernel is of course $O_l^\times$. In particular $\sharp F(I') = [(O'_l)^\times \colon O_l^\times]$. One further checks that $[I^{\mu_l}]_r = [I^{\nu_l}]_r$ if and only if $\alpha I^{\mu_l} = I^{\nu_l}$ for $\alpha \in O_l(I')^\times$. We thus get

$$\mathrm{mass}(\mathrm{Cls}_r(O)) = \sum_{[I']_r \in \mathrm{Cls}_r(O')} \sum_{I \in O_l(I')^\times \backslash F(I')} \frac{1}{w_I}.$$

Unravelling the inner sum using the observation above concludes the proof. □

In general the index $[(O'_l)^\times \colon O_l^\times]$ is determined by the so called local Eichler-symbol.[17] Since we are only interested in Eichler orders we can be more concrete. Suppose B_l splits and $O_l = \mathfrak{M}_0(l^k)$. We now take $O'_l = M_2(\mathbb{Z}_l)$. It is well known that in this case it is well known, that[18]

$$[(O'_l)^\times \colon O_l^\times] = l^k(1 + l^{-1}) = \psi(l^k).$$

With this at hand we can complete the proof of the main theorem of this section.

Proof of Theorem 5.48. We argue by induction on the number of prime divisors of M. If $M = 1$, then we are done because O is maximal. Now suppose $l \mid M$. Let O' be the Eichler order of level $M/(M, l^\infty)$. This is a superorder of O and by the results above we have

$$\mathrm{mass}(\mathrm{Cls}_r(O)) = \psi((M, l^\infty))\mathrm{mass}(\mathrm{Cls}_r(O')).$$

We conclude the prove by applying the induction hypothesis together with

$$\psi((M, l^\infty))\psi(M/(M, l^\infty)) = \psi(M).$$

□

Note that Eichler's mass formula is a very clean statement. Unfortunately it is not a straight forward task to remove the weights w_J. This can be done with the help of the theory of embedding numbers which we develop next.

[17] This should remind one of how the Kronecker-symbol turns up when relating class numbers of orders in imaginary quadratic fields to the class number of the ring of integers.

[18] This is a reincarnation of the fact $[\mathrm{SL}_2(\mathbb{Z}) \colon \Gamma_0(l^k)] = \psi(l^k)$.

Exercise 10. Let $B = \left(\frac{-2,-37}{\mathbb{Q}}\right)$ and take O as in (15). Show that O has class number 3. Further, find representatives of left O ideals and compute their norm.

Proof. In practice one can use Proposition 5.37 to find integral representatives for the class group of small norm. In our particular case one would need to find all integral ideals I with

$$1 \leq \mathrm{nr}(I) = N(I)^{\frac{1}{2}} \leq 5 < \frac{2}{\pi}\sqrt{2 \cdot 37} < 6.$$

Of course the only ideal of norm 1 is the order itself and we have $I_1 = O$. The computations for $2, 3, 4$ and 5 get slightly unpleasant. The idea however is that, since $p = 2, 3, 5$ are unramified, we can embed $O_p \hookrightarrow M_2(\mathbb{Z}_p)$ for $p = 2, 3, 5$. One then uses normal form theory to classify all possible ideals of these norms. The tricky bit is then to find the relations between them and pick convenient representatives. This is illustrated in [9][Example 17.6.3] for a different B and O.

We omit the computations but give the following set of representatives taken from [5][Example 10.1]: $\mathrm{Cls}_l = \{[I_1]_l, [I_2]_l, [I_3]_l\}$ for $I_1 = O$,

$$I_2 = (2 + 6j + 10k)\mathbb{Z} + (i + 2j + 9k)\mathbb{Z} + 12j\mathbb{Z} + 12k\mathbb{Z},$$
$$I_3 = (2 + 26j + 26k)\mathbb{Z} + (i + 2j + 13k)\mathbb{Z} + 28j\mathbb{Z} + 28k\mathbb{Z}.$$

Note that it is not hard to pass from left classes to right classes. However, these representatives don't have small norm.

□

5.5. **Embedding numbers.** Let K be a separable quadratic $\mathbb{Q}$-algebra (i.e. either a separable quadratic field extension of $\mathbb{Q}$ or $\mathbb{Q} \times \mathbb{Q}$). Embeddings from $K \hookrightarrow B$ are parametrised by $K^\times \backslash B^\times$. This can be deduced from the Skolem-Noether theorem.

We want to restrict our attention to integral embeddings. More precisely, fix an order $O \subset B$ and an order $\mathfrak{r} \subset K$. We consider embeddings $\phi\colon \mathfrak{r} \hookrightarrow O$. Such an embedding can be extended to K in the obvious way.

Definition 5.19. An embedding $\phi\colon \mathfrak{r} \hookrightarrow O$ is optimal if it satisfies

$$\phi(K) \cap O = \phi(\mathfrak{r}).$$

The set of all such embeddings is denoted by $\mathrm{Emb}(\mathfrak{r}, O)$.

This gives the following partition of the set of all embeddings:

$$\{\phi\colon \mathfrak{r} \to O\} = \bigsqcup_{\mathfrak{r}' \supset \mathfrak{r}} \mathrm{Emb}(\mathfrak{r}', O).$$

Example 5.54. Suppose $d < 0$ with $d \equiv 0, 1 \mod 4$, then $K = \mathbb{Q}(\sqrt{d})$ defines an imaginary quadratic field of discriminant $d_K = f^{-2}d$ and ring of integers $\mathcal{O}_K$. Furthermore, there is exactly one order $\mathfrak{r}_{f^2 d_K} \subset \mathcal{O}_K$ of conductor f and discriminant

d. We have

$$\{\phi\colon \mathfrak{r}_{f^2 d_K} \to O\} = \bigsqcup_{f'|f} \mathrm{Emb}(\mathfrak{r}_{f'^2 d_K}, O).$$

Lemma 5.55. *An embedding $\phi\colon \mathfrak{r} \hookrightarrow O$ is optimal if and only if $\phi_p\colon \mathfrak{r}_p \hookrightarrow O_p$ is optimal for all p.*

Proof. Follows directly from the local-to-global properties of lattices. □

Given $\gamma \in O^\times$ and an optimal embedding $\phi\colon \mathfrak{r} \hookrightarrow O$ we can form the conjugate $\alpha \mapsto \gamma^{-1}\phi(\alpha)\gamma$. We set

$$\mathrm{Emb}(\mathfrak{r}, O, O^\times) = \mathrm{Emb}(\mathfrak{r}, O)/\sim_{O^\times},$$

where $\sim_{O^\times}$ is the equivalence relation coming from conjugation with elements of $O^\times$. We set

$$m(\mathfrak{r}, O, O^\times) = \sharp\mathrm{Emb}(\mathfrak{r}, O, O^\times).$$

Lemma 5.56. *We have*

$$\mathit{Emb}(\mathfrak{r}, O) = K^\times\backslash E \text{ and } \mathit{Emb}(\mathfrak{r}, O, O^\times) = K^\times\backslash E/O^\times,$$

for

$$E = \{\beta \in B^\times\colon K \cap \beta O \beta^{-1} = \mathfrak{r}\}.$$

We have the following important result, which reduces the computation of embedding numbers to local considerations.

Proposition 5.57. *One has*

$$\sum_{[I]_r \in \mathit{Cls}_r(O)} m(\mathfrak{r}, O_l(I), O_l(I)^\times) = h(\mathfrak{r}) m(\widehat{\mathfrak{r}}, \widehat{O}, \widehat{O}^\times).$$

Here the adelic embedding number is defined by $m(\widehat{\mathfrak{r}}, \widehat{O}, \widehat{O}^\times) = \sharp\mathbb{A}_{K,f}^\times\backslash\widehat{E}/\widehat{O}^\times$, for

$$\widehat{E} = \{\beta \in \mathbb{A}_{B,f}^\times\colon \beta^{-1}\mathbb{A}_{K,f}\beta \cap \widehat{O} = \beta^{-1}\widehat{\mathfrak{r}}\beta\}.$$

Proof. We have the natural surjective map

$$K^\times\backslash\widehat{E}/\widehat{O}^\times \xrightarrow{\rho} \mathbb{A}_{K,f}^\times\backslash\widehat{E}/\widehat{O}^\times.$$

We claim that the cardinality of the fibres of ρ is precisely the class number $h(\mathfrak{r})$.[19] In general the the fibre $\rho^{-1}(\mathbb{A}_{K,f}^\times\beta\widehat{O}^\times)$ consists of

$$K^\times v\beta\widehat{O}^\times \text{ with } K^\times v \in K^\times\backslash\mathbb{A}_{K,f}^\times.$$

A short computation shows that $K^\times v\beta\widehat{O}^\times = K^\times\beta\widehat{O}^\times$ if and only if $K^\times v \subset K^\times\widehat{\mathfrak{r}}^\times$, which establishes the claim. We deduce that

$$\sharp K^\times\backslash\widehat{E}/\widehat{O}^\times = h(\mathfrak{r}) m(\widehat{r}, \widehat{O}, \widehat{O}^\times). \tag{17}$$

[19] The cleanest instance of this is $\rho^{-1}(1) = K^\times\mathbb{A}_{K,f}^\times/\widehat{\mathfrak{r}^\times} \cong \mathrm{Cls}(\mathfrak{r})$.

We now compute this cardinality in a different way. We choose representatives

$$\mathbb{A}_{B,f}^\times = \bigsqcup_{[I]\in \mathrm{Cls}_r(O)} B^\times \alpha_i \widehat{O}^\times.$$

In particular, $I = \alpha_i \widehat{O} \cap B$. To save space we write $O_I = O_l(I) = \alpha_i \widehat{O} \alpha_i^{-1} \cap B$. Finally define

$$E_I = \{\beta \in B^\times : K \cap \beta O_I \beta^{-1} = \mathfrak{r}\}.$$

For $\beta \widehat{O}^\times \in \widehat{E}/\widehat{O}^\times$ there is a unique I such that

$$B^\times \beta \widehat{O}^\times \subset B^\times \alpha_I \widehat{O}^\times.$$

In particular there is $b \in B^\times$ such that $\beta \widehat{O}^\times = (b\alpha_I \widehat{O}^\times)$ and the class $bO_I^\times$ is well defined. One checks that $b \in E_I$ and that given $b \in E_I$ one has $\beta = b\alpha_I^{-1} \in \widehat{E}$. Thus we have the bijections

$$K^\times \backslash \widehat{E}/\widehat{O}^\times \overset{K^\times\beta\widehat{O}^\times \mapsto K^\times bO_I^\times}{\longrightarrow} \bigsqcup_{[I]\in \mathrm{Cls}_r(O)} K^\times \backslash E_I/O_I^\times \to \bigsqcup_{[I]\in \mathrm{Cls}_r(O)} \mathrm{Emb}(\mathfrak{r}, O_I, O_I^\times).$$

Together with (17) this competes the proof. $\square$

This result expresses an average of embedding numbers in terms of the cardinality of an adelic (double)-quotient. There is hope that the latter can be computed place by place. Therefore we work locally and assume that $\mathfrak{r}_p = \mathbb{Z}_p[\gamma]$. Let $f_\gamma(X) = X^2 - tX + n$ be the minimal polynomial of γ and write $d = t^2 - 4n$.

Lemma 5.58. *Suppose $B_p = \mathrm{M}_2(\mathbb{Q}_p)$ and O_p is maximal. Then*

$$m(\mathfrak{r}_p, O_p, O_p^\times) = 1.$$

Proof. Because the embedding number only depends on the type of O_p we can assume that $O_p = M_2(\mathbb{Z}_p)$. Now we have a canonical optimal embedding given by

$$\gamma \mapsto \begin{pmatrix} 0 & -n \\ 1 & t \end{pmatrix}.$$

Given another embedding ψ defined by

$$\psi(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

one considers $q(x_1, x_2) = cx_1^2 + (d-a)x_1x_2 - bx_2^2$. One can see that there is $\mathbf{x} = (x_1, x_2)^t \in \mathbb{Z}_p^2$ such that $q(x_1, x_2) \in \mathbb{Z}_p^\times$. We define the matrix $\alpha = (\mathbf{x}\, \psi(\gamma)\mathbf{x})$. Since $\det(\alpha) = q(x_1, x_2)$ we have $\alpha \in \mathrm{GL}_2(\mathbb{Z}_p) = O_p^\times$. Furthermore, this matrix is constructed such that

$$\alpha^{-1}\psi(\gamma)\alpha = \begin{pmatrix} 0 & -n \\ 1 & t \end{pmatrix}.$$

We have seen that every other optimal embedding is $O^\times$-conjugate to the canonical one. This completes the proof. $\square$

We are now going to compute the local embedding numbers in some other important cases. We define the Legendre type symbol

$$\left(\frac{K}{p}\right) = \begin{cases} 1 & \text{if } K_p \cong \mathbb{Q}_p \times \mathbb{Q}_p, \\ 0 & \text{if } \mathbb{Q}_p \subset K_p \text{ is ramified}, \\ -1 & \text{if } \mathbb{Q}_p \subset K_p \text{ is unramified.} \end{cases}$$

Lemma 5.59. *Let B_p be a divison algebra and $O_p \subset B_p$ maximal. Then we have*

$$m(S_p, O_p, O_p^\times) = \begin{cases} 1 - \left(\frac{K}{p}\right) & \textit{if } \mathfrak{r}_k \textit{ is maximal}, \\ 0 & \textit{else.} \end{cases}$$

Proof. It is a well known fact, that there are only embeddings $K_p \to B_p$ if K_p is a field. Furthermore, since O_p is maximal it is the ring of integral elements in B_p. In particular, if $\mathfrak{r}_p$ is not maximal it is not integrally closed and thus can not be embedded in O in an optimal manner. Thus we take $\mathfrak{r}_p$ to be maximal.

Observe that for all $\beta \in B_p^\times$ we have $\beta O_p \beta^{-1} = O_p$, by uniqueness of the maximal order. We conclude that $E = B_p^\times$. By Corollary 5.20 we have $O_p = S \oplus Sj$, where S is the maximal order in the unique unramified quadratic extension F_p of $\mathbb{Q}_p$ and j is an element in B with $\mathrm{nr}(j) = p$.

If K_p is unramified, then $K_p = F_p$ and $\mathfrak{r}_p = S$. In this case we have two optimal embeddings, since conjugation by j normalises K_p.

If K_p is ramified, then $K_p = \mathbb{Q}_p[j]$ and this determines the only optimal embedding. □

This lemma can be nicely rephrased in terms of the modified Kronecker-symbol defined in (13).

Corollary 5.60. *Let $\mathbf{r} \subset K$ be an order in an imaginary quadratic field K of discriminant $d_{\mathfrak{r}}$,. Further let B_p be non-split and let $O_p \subset B_p$ be the maximal order. Then*

$$m(S_p, O_p, O_p^\times) = 1 - \left\{\frac{d_{\mathfrak{r}}}{p}\right\}.$$

Lemma 5.61. *Let B_p be split and $O_p \cong \mathfrak{M}_0(p)$. Further assume that $\mathfrak{r} \subset K$ is an order of discriminant $d_{\mathfrak{r}}$. Then we have*

$$m(\mathfrak{r}_p, O_p, O_p^\times) = 1 + \left\{\frac{d_{\mathfrak{r}}}{p}\right\}.$$

Proof. We will only sketch this proof. A reincarnation of Atkin-Lehner theory tells us that (for $k > 0$)

$$N_{B_p^\times}(O_p) = \{\beta \in \mathrm{GL}_2(\mathbb{Q}_p) \colon \beta \mathfrak{M}_0(p^k)\beta^{-1} = \mathfrak{M}(p^k)\} = \langle \mathbb{Q}_p \times O_p^\times, W_{p_k} \rangle,$$

for $W_{p^k} = \begin{pmatrix} 0 & 1 \\ p^k & 0 \end{pmatrix}$. Given an optimal embedding ϕ we define the **A-L-conjugate embedding**[20] by

$$\phi^{A-L}(\alpha) = W_{p^k}^{-1}\phi(\alpha)W_{p^k}.$$

Further we call an embedding **normalised** and **associated** to x if

$$\phi(\gamma) = \begin{pmatrix} x & 1 \\ -f_\gamma(x) & t-x \end{pmatrix}.$$

The proof now roughly proceeds as follows.

First, one shows that either the class of ϕ or the class ϕ^{A-L} in $\mathrm{Emb}(\mathfrak{r}_p, O_p, O_p^\times)$ is represented by a normalised embedding.

Second, one investigates when normalised embeddings are conjugate by $O_p^\times$. One finds

(1) Two normalised embeddings ϕ and ϕ' are $O_p^\times$ conjugate if and only if $x \equiv x' \bmod p^k$;
(2) If $d \in \mathbb{Z}_p^\times$, then ϕ^{A-L} and ϕ' are $O_p^\times$ conjugate if and only if $x' \equiv t - x \bmod p^k$;
(3) If $d \in p\mathbb{Z}_p$, then ϕ^{A-L} and ϕ' are $O_p^\times$ conjugate if and only if $x' \equiv t - x \bmod p^k$ and $f_\gamma(x) \not\equiv 0 \bmod p^{k+1}$;

Third, we parametrise the set of optimal embeddings up to conjugation by $O_p^\times$ terms of normalised embeddings and their Atkin-Lehner conjugates. Using the list above this leads to

$$m(\mathfrak{r}_p, O_p, O_p^\times) = \sharp M(k) + \delta_{p|d}\sharp[\mathrm{img}(M(k+1) \to \mathbb{Z}_p/p^k\mathbb{Z}_p)],$$

for

$$M(e) = \{x \in \mathbb{Z}_p/p^e\mathbb{Z}_p \colon f_\gamma(x) \equiv 0 \bmod p^e\}.$$

Finally, one counts elements in $M(e)$ using the arithmetic of $\mathbb{Z}_p$. □

Corollary 5.62. *Altogether we have seen that, if O is hereditary and $\mathfrak{r} \subset K$ is an order of discriminant $d_\mathfrak{r}$, then*

$$m(\widehat{\mathfrak{r}}, \widehat{O}, \widehat{O}^\times) = \prod_{p|\mathrm{disc}(B)} \left(1 - \left\{\frac{d_\mathfrak{r}}{p}\right\}\right) \prod_{p|\frac{\mathrm{disc}(O)}{\mathrm{disc}(B)^2}} \left(1 + \left\{\frac{d_\mathfrak{r}}{p}\right\}\right).$$

If $\mathfrak{r}$ is maximal, then the modified Kronecker symbols reduce to classical Legendre symbols.

We will encounter these notions again when we are computing the traces of Brandt matrices. On a similar note one can use embedding numbers to compute the class number on the nose. This is reflected in the following theorem.

[20]Disclaimer: No standard terminology!

Theorem 5.63 (Eichler's class number formula)**.** *Let B be a definite quaternion algebra over $\mathbb{Q}$ and $O \subset B$ be a $(\mathbb{Z})$-order. Then we have*

$$\sharp Cls_r(O) = mass(Cls_r(O)) + \frac{1}{2}\sum_{q\geq 2}\sum_{\sharp\mathfrak{r}^\times/\{\pm 1\}=q} h(\mathfrak{r})m(\widehat{\mathfrak{r}},\widehat{O},\widehat{O}^\times).$$

We postpone the proof for now. A special case of this is the following.

Theorem 5.64. *Let B be a definite Quaternion algebra over $\mathbb{Q}$ and let O be an Eichler order with $\mathrm{disc}(O) = M^2D^2$, where $D = \mathrm{disc}(B)$ and $(M, D) = 1$. Then*

$$\sharp Cls_r(O) = \frac{\varphi(D)\psi(M)}{12} + \frac{\epsilon_2}{4} + \frac{\epsilon_3}{3},$$

for

$$\epsilon_2 = \begin{cases} \prod_{p\mid D}\left(1-\left(\frac{-4}{p}\right)\right)\prod_{p\mid M}\left(1+\left(\frac{-4}{p}\right)\right) & \text{if } 4\nmid \mathrm{disc}(O), \\ 0 & \text{else,}\end{cases} \tag{18}$$

and

$$\epsilon_2 = \begin{cases} \prod_{p\mid D}\left(1-\left(\frac{-3}{p}\right)\right)\prod_{p\mid M}\left(1+\left(\frac{-3}{p}\right)\right) & \text{if } 9\nmid \mathrm{disc}(O), \\ 0 & \text{else.}\end{cases} \tag{19}$$

One can also compute Type numbers using similar machinery. An example is the following result.

Theorem 5.65 (Deuring)**.** *Let B be a quaternion algebra with $\mathrm{disc}(B) = p \geq 5$ and let O be a maximal order. Then*

$$\sharp Typ(O) = \frac{\sharp Cls_r(O)}{2} + \frac{\sharp Cl(\mathbb{Q}(\sqrt{-p}))}{4}\cdot\begin{cases} 1 & \text{if } p\equiv 1 \text{ mod } 4,\\ 4 & \text{if } p\equiv 3 \text{ mod } 8,\\ 2 & \text{if } p\equiv 7 \text{ mod } 8.\end{cases}$$

Exercise 11. Compute $\mathrm{mass}(\mathrm{Cls}_r(O))$, $\sharp\mathrm{Cls}_r(O)$ and $\sharp\mathrm{Typ}(O)$ for O as in (15).

Proof. Of course we have

$$\mathrm{mass}(\mathrm{Cls}_r(O)) = \frac{\varphi(37)}{12} = 3.$$

Further, by Eichler's class number formula we find

$$\sharp\mathrm{Cls}_r(O) = 3 + \frac{1}{4}\left(1-\left(\frac{-4}{37}\right)\right) + \frac{1}{3}\left(1-\left(\frac{-3}{37}\right)\right) = 3.$$

In particular, writing $\mathrm{Cls}_r(O) = \{[I_1],[I_2],[I_3]\}$ we find that $O_l(I_i)^\times = \{\pm 1\}$ for $1 \leq i \leq 3$. Note that $\mathrm{Cls}(\mathbb{Q}(\sqrt{-37})) = \mathbb{Z}/2\mathbb{Z}$, so that Deuring's result yields

$$\sharp\mathrm{Typ}(O) = 2.$$

□

Exercise 12. Fill in the details in the proof of Lemma 4.28 using the results of this section.

5.6. **Brandt Matrices.** Let B be a definite quaternion algebra over $\mathbb{Q}$ and let $O \subset B$ be an hereditary order. In the previous section we have seen that $h = \sharp \mathrm{Cls}_r(O) < \infty$. Further we can fix (integral) representatives (with small norm) $I_1, \ldots, I_h$ such that

$$\mathrm{Cls}_r(O) = \{[I_1]_r, \ldots, [I_h]_r\}.$$

Definition 5.20. We define the **Brandt Matrix** $T(n) \in M_h(\mathbb{Z})$ by

$$T(n)_{ij} = \sharp\{J \subset I_j \colon \mathrm{nr}(J) = n \cdot \mathrm{nr}(I_j) \text{ and } [J]_r = [I_i]_r\}.$$

Remark 5.66. For $(n, \mathrm{disc}(O)) \neq 1$ this definition usually appears in slightly different form in order to make the Hecke-relations work for $p \mid \mathrm{disc}(O)$. This modification would of course also effect the right hand side of Eichler's trace formula for Brandt matrices as soon as $(n, \mathrm{disc}(O)) \neq 1$. However, since we will not really need these ramified Hecke-operators we will stick to this easier definition.

Let $M_2(O) = \{\mathrm{Cls}_r \to \mathbb{Z}\}$. This is a free $\mathbb{Z}$-module with (canonical basis) $\mathcal{B} = \{\delta_{[I_i]_r} \colon 1 \leq i \leq h\}$. In this basis the Brandt-matrix defines a $\mathbb{Z}$-linear map $T(n) \colon M_2(O) \to M_2(O)$, which we call the nth Hecke operator. This operator can be described by

$$[T(n)f]([I]_r) = \sum_{\substack{J \subset I, \\ \mathrm{nr}(J) = n \cdot \mathrm{nr}(I)}} f([J]_r).$$

Lemma 5.67. *Let $q_i = nr(I_i)$, $O_i = O_l(I_i)$ and $w_i = \sharp O_i^\times/\{\pm 1\}$. Then*

$$T(n)_{ij} = \frac{1}{2w_i} \sharp\{\alpha \in I_j I_i^{-1} \colon nr(\alpha) = \frac{q_j n}{q_i}\}.$$

Proof. We observe that $\alpha I_i = J \subset I_j$ and $\mathrm{nr}(J) = n \cdot \mathrm{nr}(I_j)$ if and only if

$$\alpha \in I_j I_i^{-1} \text{ and } \mathrm{nr}(\alpha) q_i = n q_j.$$

We observe that α is uniquely determined up to right multiplication by $\mu \in O_i^\times$. $\square$

Remark 5.68. Note that this reduces the computation of the entries $T(n)_{ij}$ of the nth Brandt matrix to a lattice point enumeration. Indeed, we have a quadratic form

$$Q_{ij} \colon I_j I_i^{-1} \to \mathbb{Z},\ \alpha \mapsto \mathrm{nr}(\alpha) \frac{q_j}{q_i}.$$

The entries are essentially given by the representation numbers $a(n; Q_{ij})$.

Proposition 5.69. *The Brandt matrices have the following properties.*

(1) *The sums $\sum_{i=1}^h T(n)_{ij}$ are constant in j. Furthermore, if $(n, \mathrm{disc}(O)) = 1$, then we have*

$$\sum_{i=1}^h T(n)_{ij} = \sigma_1(n).$$

(2) *If* $(m,n)=1$, *then*

$$T(mn)=T(n)T(m).$$

Proof. We first prove the first point. To do so let $f\equiv 1$ be the constant one function. Observe that

$$\sum_i T(n)_{ij}=[T(n)f](I_j)=\sum_{\substack{J\subset I_j\\ \mathrm{nr}(J)=n\cdot\mathrm{nr}(I_j)}}1=\sharp\{J\subset O_l(I_j)\colon \mathrm{nr}(J)=n\}=a_n(O_l(I_j)).$$

Since $O_l(I_j)\in\mathrm{Gen}(O)$ for $j=1,\dots,h$ the right hand side is independent of j. Furthermore, if $(n,\mathrm{disc}(O))=1$ we have computed it before.

Due to the definition of matrix multiplication we have to show that

$$T(nm)_{ij}=\sum_{k=1}^h T(n)_{ik}T(m)_{kj}.$$

Given an ideal J contributing to $T(nm)_{ij}$ we have $\mathrm{nr}(JI_j^{-1})=nm$. Since $(n,m)=1$ we can apply Lemma 5.45. This determines a unique ideal J_1 such that $\mathrm{nr}(J_1)=n$ and $\mathrm{nr}(JI_j^{-1}J_1^{-1})=n$. From this decomposition one can construct a bijection between ideals J contributing to the count of $T(nm)$ and tuples of ideals $(J_1I_k, JI_j^{-1}J_1^{-1}I_j)$ contributing the the right hand count. □

Proposition 5.70. *We have*

$$T(p^{r+2})=T(p^{r+1})T(p)-pT(p^r).$$

Proof. We would like to apply a similar tactic as previously and argue by uniquely factoring ideals contributing to $T(p^{r+2})$. However, this is not possible without slightly modifying the argument.

We call an ideal I **primitive** if it can not be written as $I=aI'$ for an integral Ideal I' and $a\in\mathbb{Z}$. We now decompose

$$T(p^{r+2})=T_{prim}(p^{r+2})+T_{imprim}(p^{r+2})$$

accordingly. The upshot is, that for primitive ideals unique factorisation as earlier essentially works.

We first observe that $T_{imprim}(p^{r+2})=T(p^{r-1})$. Now we look at

$$[T(p^{r+1})T(p)]_{ij}=\sum_{k=1}^h T(p^{r+1})_{ik}T(p)_{kj}.$$

We can understand this as counting products $J_r'J'$ with J_r' being a invertible-O_i,O_k-ideal with $\mathrm{nr}(J_r')=p^r$ and J' being a invertible-O_k,O_j-ideal with $\mathrm{nr}(J')=p$. If the product is primitive then it uniquely determines J_r and J' and it contributes to the count of $T_{prim}(p^{r+2})_{ij}$. So let us assume that $J_r'J'$ is imprimitive. It can be shown, that there are exactly as many configurations giving $J_r'J' = \tilde{J}_r'\tilde{J}'$ as there are right-O-ideals of norm p. Since there are $p+1$ such ideals we get

$$T(p^{r+1})T(p) = T_{prim}(p^{r+2}) + (p+1)T_{imprim}(p^{r+2})$$
$$= T(p^{r+2}) + pT_{imprim}(p^{r+2}) = T(p^{r+2}) + pT(p^r).$$

□

We now define a bilinear form $\langle\cdot,\cdot\rangle\colon M_2(O)\times M_2(O)\to\mathbb{Z}$ which is symmetric and non-degenerate. On the basis $\mathcal{B}$ this bilinear form is given by

$$\langle \delta_{[I_i]_r}, \delta_{[I_j]_r}\rangle = w_i \cdot \delta_{i,j}.$$

Proposition 5.71. *For* $(n, \mathrm{disc}(O)) = 1$ *the operator* $T(n)$ *is self adjoint with respect to* $\langle\cdot,\cdot\rangle$. *In symbols,* $T(n)^* = T(n)$.

Proof. Let $W = \mathrm{diag}(w_1, \dots, w_h)$. Using the basis $\mathcal{B}$ we can identify $M_2(O)$ with $\mathbb{Z}^h$. Writing elements as row vectors we have $\langle x, y\rangle = xWy^t$. Further the Brandt matrices act by right multiplication and the adjoint, defined by $\langle xT(n), y\rangle = \langle x, T(n)^*y\rangle$, is given by

$$T(n)^* = W^{-1}T(n)^tW.$$

We define $A(n)_{ij} = \{\alpha\in I_jI_i^{-1}\colon \mathrm{nr}(\alpha) = n\frac{q_j}{q_i}\}$. Thus,

$$WT(n) = (\frac{1}{2}\sharp A(n)_{ij})_{i,j}.$$

Our goal is to construct a bijection from $A(n)_{ij}$ to $A(n)_{ji}$. This implies that

$$WT(n) = (WT(n))^t = T(n)^tW$$

and $T(n)^* = W^{-1}T(n)^tW = T(n)$ follows straight away.

The map is defined by

$$A(n)_{ij}\to A(n)_{ji}, \alpha\mapsto n\alpha^{-1}.$$

If this map is well defined its obviously bijective. So let us start by checking

$$\mathrm{nr}(n\alpha^{-1}) = n^2n^{-1}q_iq_j^{-1} = n\frac{q_i}{q_j}$$

as needed. Further we compute that

$$\overline{\alpha}\in\overline{I_jI_i^{-1}} = \overline{I_i^{-1}}\,\overline{I_j} = \frac{q_j}{q_i}I_iI_j^{-1}.$$

Thus,

$$n\alpha^{-1} = \frac{n}{\mathrm{nr}(\alpha)}\overline{\alpha}\in\underbrace{\frac{nq_j}{\mathrm{nr}(\alpha)q_i}}_{=1}I_iI_j^{-1}.$$

□

Definition 5.21. Let $\mathbf{T}(O)$ be the subring of $M_h(\mathbb{Z})$ generated by $T(n)$ with $(n, \mathrm{disc}(O)) = 1$. We call this the **unramified Hecke-algebra** of O.

Corollary 5.72. *The space $M_2(O)$ is a (free) finite dimensional $\mathbf{T}(O)$ module featuring a basis of simultaneous $\mathbf{T}(O)$ eigenfunctions.*

Proof. This follows since $\mathbf{T}(O)$ is generated from a family of commuting operators that are self adjoint with respect to $\langle\cdot,\cdot\rangle$. $\square$

Corollary 5.73. *The ring $\mathbf{T}(O)$ is a commutative semisimple $\mathbb{Z}$-algebra.*

Exercise 13. Compute the Brandt matrices $T(n)$ with $n = 1, 2, 3$ for the order O defined in (15).

5.7. **The Eichler-Brandt trace formula.** Let B be a definite quaternion algebra over $\mathbb{Q}$ and let O be a hereditary ($\mathbb{Z}$)-order with $\mathrm{disc}(O)^{\frac{1}{2}} = DM$ for $D = \mathrm{disc}(B)$ and M square-free with $(D, M) = 1$.

Theorem 5.74 (Eichler's trace formula IIa)**.** *For $(n, \mathrm{disc}(O)) = 1$ we have*

$$Tr(T(n)) = \frac{1}{2} \sum_{\substack{t\in\mathbb{Z} \\ t^2-4n<0}} \sum_{f^2|t^2-4n} h_O\left(\frac{t^2-4n}{f^2}\right) + \delta_{\sqrt{n}\in\mathbb{Z}} \frac{\varphi(D)M\psi(M)}{12}.$$

with

$$h_O(d) = \frac{h(S_d)}{\sharp(S_d^\times/\{\pm 1\})} \prod_{p|\mathrm{disc}(O)} m([S_d]_p, O_p, O_p^\times).$$

We call the numbers $h_O(d)$ modified Hurwitz class numbers.

Proof. Note that

$$\mathrm{Tr}(T(n)) = \sum_i T(n)_{ii}.$$

We write

$$2w_i T(n)_{ii} = \sum_{t\in\mathbb{Z}} \sharp\{\alpha \in O_i \colon \mathrm{nr}(\alpha) = n,\ \mathrm{tr}(\alpha) = t\}.$$

If and only if n is a perfect square we have the contribution $\alpha = \pm\sqrt{n} \in \mathbb{Q}$ to the count. On the other hand, given any such $\alpha \notin \mathbb{Q}$ contributing to the count we consider the ring $\mathbb{Z}[\alpha] \cong \mathbb{Z}[X]/(X^2 - tX + n)$. This is an order of discriminant $t^2 - 4n$ in $\mathbb{Q}(\alpha)$. since any embedding of $\mathbb{Z}[\alpha]$ is uniquely determined by its value on α we obtain

$$\sharp\{\alpha \in O_i \colon \mathrm{nr}(\alpha) = n,\ \mathrm{tr}(\alpha) = t\} - 2\delta_{n=\square} = 2\sharp\{\phi \colon \mathbb{Z}[\alpha] \to O_i\} = \sum_{\mathbb{Z}[\alpha]\subset S} 2\sharp\mathrm{Emb}(S, O_i).$$

Note that $\sharp\mathrm{Emb}(S, O_i) = \frac{w_i}{2\sharp(S^\times/\{\pm1\})} m(S, O_i, O_i^\times)$. Further all orders in quadratic fields are determined by their discriminant and real quadratic fields can not be embedded into definite quaternion algebras. We conclude

$$\mathrm{Tr}(T(n)) = \frac{1}{2} \sum_{\substack{t\in\mathbb{Z}, \\ t^2-4n<0}} \sum_{\substack{f\geq 1, \\ f^2|t^2-4n}} \frac{1}{\sharp(S^\times_{\frac{t^2-4n}{f^2}}/\{\pm1\})} \sum_{i=1}^{h} m(S_{\frac{t^2-4n}{f^2}}, O_i, O_i^\times) + \delta_{n=\square}\mathrm{mass}(\mathrm{Cls}_r(O)).$$

We conclude by Proposition 5.57. $\square$

Remark 5.75. According to the explicit computations of local embedding numbers we can compute the Hurwitz class numbers more explicitly in terms of modified Kronecker-symbols. For a hereditary order O, we get

$$h_O(d) = \frac{h(S_d)}{\sharp(S_d^\times/\{\pm 1\})} \prod_{p \mid \frac{\mathrm{disc}(O)}{\mathrm{disc}(B)^2}} \left(1 + \left\{\frac{d}{p}\right\}\right) \prod_{p \mid \mathrm{disc}(B)} \left(1 - \left\{\frac{d}{p}\right\}\right). \tag{20}$$

Exercise 14. Prove Theorem 5.63 using the trace formula for Brandt matrices.

5.8. **Extending the Brandt matrix to non-trivial representations.** In this section we have to extend the definition of the Brandt matrix by a symmetric kth-power. This is necessary to deal with the basis problem for arbitrary (even) weight.

We start by making the following inclusion

$$\rho\colon B^\times \longrightarrow \mathrm{GL}_2(\mathbb{C}),\, t + ix + jy + kz \mapsto \begin{pmatrix} t + \sqrt{-a}xi_{\mathbb{C}} & \sqrt{-b}y + \sqrt{ab}zi_{\mathbb{C}} \\ -\sqrt{-b}y + \sqrt{ab}zi_{\mathbb{C}} & t - \sqrt{-a}xi_{\mathbb{C}} \end{pmatrix}.$$

Note that since B is definite we have $a, b < 0$ and the square roots are all real. Further note that this inclusion satisfies

$$\mathrm{tr}(\alpha) = \mathrm{tr}(\rho(\alpha)) \text{ and } \mathrm{nr}(\alpha) = \det(\rho(\alpha)).$$

Further we have the canonical map $\mathrm{sym}^k\colon \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{k+1}(\mathbb{C})$ given by lifting the standard representation of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$ to $\mathrm{Sym}^k(\mathbb{C}^2)$. Combining these two maps we obtain

$$\rho_k = \mathrm{sym}^k \circ \rho\colon B^\times \to \mathrm{GL}_{k+1}(\mathbb{C}).$$

We collect two important results.

Lemma 5.76. *The functions defined by*

$$\rho_k(x_1 + x_2 i + x_3 j + x_4 k) = (P_{ij}(x_1, x_2, x_3, x_4))_{0 \le i,j \le k}$$

are harmonic polynomials of degree k.

Proof. This can be reduced to well known facts from classical representation theory of compact Lie-groups. $\square$

Lemma 5.77. *Suppose $\alpha \in B^\times$ is such that $nr(\alpha) = n$ and $tr(\alpha) = t$. Then*

$$\mathrm{tr}(\rho_k(\alpha)) = \frac{\lambda_n^+(t)^{k+1} - \lambda_n^-(t)^{k+1}}{\lambda_n^+(t) - \lambda_n^-(t)},$$

where $\lambda_n^+(t), \lambda_n^-(t)$ are the two solutions of $X^2 - nX + n$.

Proof. This can be computed by diagonalising $\rho(\alpha)$. $\square$

For $(n, \mathrm{disc}(O)) = 1$ we define the n-Brandt-matrix of weight k by

$$T_k(n) = (M_{ij}^{(k)}(n))_{1\le i,j\le h} \text{ for } M_{ij}^{(k)}(n) = \frac{1}{2w_i}\sum \rho_k(\alpha) \in M_{k+1}(\mathbb{Q}).$$

One can similarly define $T_k(n)$ for $(n, \mathrm{disc}(O)) \neq 1$. However in this case a slight modification is to ensure the matrices satisfy a Hecke-like-relation. We will omit these technicalities.

We define the $\mathbb{Z}$-algebra

$$\mathbf{T}_k(O) = \langle T_k(n) \colon (n, \mathrm{disc}(O)) = 1\rangle_{\mathbb{Z}}.$$

Theorem 5.78. *The $\mathbb{Z}$-algebra $\mathbf{T}_k(O)$ is commutative and semisimple. Furthermore the Brandt-matrices satisfy the Hecke-relation*

$$T(mp^{l+1}) = T(mp^l)T(p) - \delta_{p\nmid \mathrm{disc}(O)} \cdot \delta_{l>0} \cdot p^{k+1}T(mp^{l-1}) \tag{21}$$

for all primes p and $(m,p)=1$.

Proof. We skip the proof and hope that the underlying principles are appropriately illustrated by the $k=0$ case discussed above. Note that the Hecke-relation for primes $p \mid \mathrm{disc}(O)$ relies on a suitable definition of the Brandt matrices $T(p)$ for such p, which has not been discussed. ☐

Theorem 5.79 (Eichler's trace formula IIb)**.** *Let O be a hereditary order of level M, k even and $(n, \mathrm{disc}(O)) = 1$. Then*

$$Tr(T_k(n)) = \delta_{n=\square}(k+1)n^{\frac{k}{2}}\frac{M\psi(M)\varphi(D)}{12} + \frac{1}{2}\sum_{\substack{t\in\mathbb{Z}\\ t^2-4n<0}} \frac{\lambda_n^+(t)^{k+1} - \lambda_n^-(t)^{k+1}}{\lambda_n^+(t) - \lambda_n^-(t)}$$

$$\cdot \sum_{f^2\mid t^2-4n} \frac{h(S_{\frac{t^2-4n}{f^2}})}{\sharp(S^\times_{\frac{t^2-4n}{f^2}}/\{\pm1\})} \prod_{p\mid \frac{\mathrm{disc}(O)}{\mathrm{disc}(B)^2}}\left(1+\left\{\frac{\frac{t^2-4n}{f^2}}{p}\right\}\right)\prod_{p\mid \mathrm{disc}(B)}\left(1-\left\{\frac{\frac{t^2-4n}{f^2}}{p}\right\}\right),$$

with the usual notation.

Proof. Note that $\mathrm{tr}(T_k(n)) = \sum_{i=1}^h \mathrm{tr}(M_{ii}^{(k)}(n))$. We proceed as before and compute

$$\begin{aligned} 2w_i\,\mathrm{tr}(M_{ii}^{(k)}(n)) &= \sum_{t\in\mathbb{Z}}\sum_{\substack{\alpha\in O_i,\\ \mathrm{nr}(\alpha)=n,\\ \mathrm{tr}(\alpha)=t}} \mathrm{tr}(\rho_k(\alpha)) \\ &= \sum_{t\in\mathbb{Z}}\sum_{\substack{\alpha\in O_i,\\ \mathrm{nr}(\alpha)=n,\\ \mathrm{tr}(\alpha)=t}} \frac{\lambda_n^+(t)^{k+1} - \lambda_n^-(t)^{k+1}}{\lambda_n^+(t) - \lambda_n^-(t)}, \end{aligned}$$

where we applied Lemma 5.77. The terms of the inner sum are independent of α, so that we can proceed as in the proof of Theorem 5.74 and use (20) to explicate the Hurwitz class numbers. Note that if n is a square the contribution of $\alpha = \pm\sqrt{n} \in \mathbb{Q}$ is weighted by

$$\mathrm{tr}(\rho_k(\pm\sqrt{n})) = (k+1)n^{\frac{k}{2}}.$$

$\square$

Theorem 5.80. *If $k > 0$, then the series*

$$\theta(z; O, k)_{ij} = \sum_{n=1}^{\infty} T_k(n)_{ij} e(nz), \text{ for } 1 \leq i, j \leq (k+1)h,$$

are generalised theta series and satisfy

$$\theta(z; O, k)_{ij} \in S_{k+2}(\mathrm{disc}(O)^{\frac{1}{2}}, 1).$$

Proof. By writing out the definition of the coefficients $T_k(n)_{ij}$ the fact that the series are indeed generalised θ-series follows from Lemma 5.76. The result then follows from Proposition 3.7 after identifying the correct level. $\square$

We will refer to the matrix $\theta(z; O, k) = (\theta(z; O, k)_{ij})_{ij}$ of theta series as the **Brandt theta series**.

Exercise 15. Explain why Theorem 5.80 fails for $k = 0$. Further modify the original Brandt matrices to fix this problem.

6. The basis problem

Recall that our goal was to decompose the $\mathbf{T}(N)$ -module $S_k(N,\mathrm{Id})$ into submodules, which consist of (quaternionic) theta series and carry a intrinsic Hecke-action.

Let us first introduce some notation. Given a space of functions $V \subset \{f\colon \mathbb{H} \to \mathbb{C}\}$ we define

$$V^K = \{z \mapsto f(Kz)\colon f \in V\}.$$

Further given two positive square-free integers M and H we write $O_{M,H}$ for a hereditary order of level M in the quaternion algebra B of discriminant H. Note that we will always ensure that H has an odd number of prime divisors ($\mu(H) = -1$) so that B is definite.

For some $1 \leq j_0 \leq (k+1)h$ we define the $\mathbb{C}$-vector space

$$\Theta_k(O_{M,H}) = \langle \theta(z, O_{M,H}, k)_{ij_0}\colon 1 \leq i \leq (k+1)h\rangle_{\mathbb{C}}.$$

This is the space of theta series spanned by the entries of one (the j_0th) column of the Brandt theta series. Note that the dimension of $\Theta_k(O_{M,H})$ is in general not $(k+1)h$. In other words, the theta series $\theta(z, O_{M,D}, k)_{ij_0}$ are not linearly independent.

Lemma 6.1. *For $(p,K) = 1$ we have*

$$T_n\theta(K\cdot, O_{M,H}, k)_{ij_0} = \sum_{l=1}^{(k+1)h} T_k(n)_{il}\theta(K\cdot, O_{M,H}, k)_{lj_0}.$$

Here T_n is the nth Hecke-operator on $S_{k+2}(KMH, Id)$ and $T_k(n)$ is the nth Brandt-matrix of weight k associated to $O_{M,H}$.

Proof. Since the Brandt matrices $T(n)$ and the Hecke-operators satisfy the same (Hecke)-relations it is enough to verify the statement for $n = p$ prime. We will only consider primes $(p, KMH) = 1$ and omit the details in the remaining cases.

We start by noting that T_p acts as follows:

$$[T_p\theta(K\cdot, O_{M,H}, k)_{ij_0}](z) = \sum_{m\geq 1}\left[\delta_{p|m}p^{k+1}T_k(\frac{m}{p})_{ij_0} + T_k(pm)_{ij_0}\right] e(mKz).$$

From (21) we deduce that $T_k(pm) + \delta_{p|m}p^{k+1}T_k(\frac{m}{p}) = T_k(p)T_k(m)$. Looking at the entry ij_0 reveals

$$\begin{aligned}[T_p\theta(K\cdot, O_{M,H}, k)_{ij_0}](z) &= \sum_{m\geq 1}\left[T_k(p)T_k(m)\right]_{ij_0} e(mKz)\\ &= \sum_{l=1}^{(k+1)h} T_k(p)_{ik}\sum_{m\geq 1} T_k(m)_{kj_0}e(mKz).\end{aligned}$$

This obviously concludes the proof. $\square$

This lemma ensures that the spaces $\Theta_k(O_{M,H})^K$ are $\mathbf{T}(MHK)$ sub-modules of $S_{k+2}(MHK, \mathrm{Id})$. These subspaces are exactly the pieces we want to decompose our space in. Note that we will not be able to cover forms that are lifted from level 1, since the quaternionic theta functions always feature at least the discriminant of the underlying quaternion algebra in their level. Thus we will not be able to avoid the subspace $S_k(1, \mathrm{Id})$ itself in our decomposition. We are now ready to state the main theorem of this course.

Theorem 6.2. *For square-free N, $k > 2$ even and $p \mid N$ we have*

$$S_k(N, Id) = \Theta_{k-2}(O_{\frac{N}{p},p}) \oplus S_k\left(\frac{N}{p}, Id\right) \oplus S_k\left(\frac{N}{p}, Id\right)^p. \tag{22}$$

Iterating this theorem yields the following corollary.

Corollary 6.3. *Let $N = p_1 \cdot \ldots \cdot p_r$, then we have*

$$S_k(N, Id) = \left[\bigoplus_{i=1}^{r} \bigoplus_{K \mid \frac{N}{\prod_{j=1}^{i-1} p_j}} \Theta(O_{\prod_{j=i+1}^{r} p_j, p_i})^K\right] \oplus \left[\bigoplus_{K \mid N} S_k(1, Id)\right].$$

Remark 6.4. Let us make the following computation as a reality check. Using (14) we find

$$\begin{aligned}
&\dim S_k(N, \mathrm{Id}) - 2 \dim S_k(\frac{N}{p}, \mathrm{Id}) = \operatorname{Tr} T_1|_{S_k(N,\mathrm{Id})} - 2 \operatorname{Tr} T_1|_{S_k(\frac{N}{p},\mathrm{Id})} \\
&= \frac{k-1}{12} \cdot \frac{N}{p} \cdot \psi(\frac{N}{p}) \cdot \varphi(p) - \frac{(-1)^{\frac{k}{2}}}{4} \prod_{p' \mid \frac{N}{p}} \left(1 + \left(\frac{-4}{p}\right)\right) \cdot \left(1 - \left(\frac{-4}{p'}\right)\right) \\
&\quad + \frac{(-1)^{\eta}}{3} \delta_{3 \nmid k-1} \prod_{p' \mid \frac{N}{p}} \left(1 + \left(\frac{-3}{p'}\right)\right) \cdot \left(1 - \left(\frac{-3}{p}\right)\right) \\
&= \operatorname{Tr} T_{k-2}(1).
\end{aligned}$$

Where we used Theorem 5.79 in the last step. Note that using the trace formula for $k = 2$ one obtains

$$\dim S_k(N, \mathrm{Id}) - 2 \dim S_k(\frac{N}{p}, \mathrm{Id}) = \sharp \mathrm{Cls}(O_{\frac{N}{p},p}) - 1 = \operatorname{tr} T_0(1) - 1.$$

This is another incarnation of the little correction that has to be made for $k = 2$.

The proof of our main theorem will heavily rely on the comparison of traces. The trace of T_n, at least for $(n, N) = 1$, on $S_k(N, \mathrm{Id})$ is given by Theorem 4.42. The following lemma allows us to compute the trace of $T_n|_{S_k(\frac{N}{p},\mathrm{Id}) \oplus S_k(\frac{N}{p},\mathrm{Id})}$ using Theorem 4.42 for $N' = \frac{N}{p}$.

Lemma 6.5. *Let T_n be the nth Hecke-operator acting on $S_k(N)$. Fix a basis $\mathcal{B}$ for $S_k(\frac{N}{p}, Id)$, then $\mathcal{B}^p = \{\phi(p\cdot)\colon \phi \in \mathcal{B}\}$ is obviously a basis for $S_k(\frac{N}{p}, Id)^p$. Let $\mathfrak{T}_n$ be the matrix representing the nth Hecke-operator acting on $S_k(\frac{N}{p}, Id)$ with respect to the basis $\mathcal{B}$. Then*

$$M_{\mathcal{B}\cup\mathcal{B}^p}(T_n|_{S_k(\frac{N}{p},Id)\oplus S_k(\frac{N}{p},Id)}) = \begin{cases} \begin{pmatrix} \mathfrak{T}_n & 0 \\ 0 & \mathfrak{T}_n \end{pmatrix} & \text{if } (n,p)=1, \\ \begin{pmatrix} -\delta_{p^2|n} p^{k-1}\mathfrak{T}_{\frac{n}{p^2}} & -p^{k-1}\mathfrak{T}_{\frac{n}{p}} \\ \mathfrak{T}_{\frac{n}{p}} & \mathfrak{T}_n \end{pmatrix} & \text{else.} \end{cases}$$

Proof. This is easily verified taking the definition of the Hecke-operators for different levels into account. □

We will only need this for $(n, N) = 1$, in which case we get

$$\mathrm{Tr}(T_n|_{S_k(\frac{N}{p},\mathrm{Id})\oplus S_k(\frac{N}{p},\mathrm{Id})}) = 2\,\mathrm{Tr}(T'_n),$$

where T'_n is the nth Hecke operator on $S_k(N', \mathrm{Id})$. We replace the action of the Hecke-operator on $\Theta_{k-2}(O_{\frac{N}{p},p})$ by the nth Brandt-matrix $T_{k-2}(n)$ of weight $k-2$, for which we can compute the trace using Theorem 5.79.

Proof of Theorem 6.2. Note that both sides of (22) are (finite dimensional) representations of the semisimple ring $\mathbf{T}(N)$. Indeed, this is obvious for the left hand side. On the right hand side we let $T(n)$ act by $T_{k-2}(n) \oplus T'_n \oplus T'_n$, which is well defined since $T_{k-2}(n)$, T'_n and T_n satisfy the same relations for $(n, N) = 1$ (see Theorem 5.78). The theorem will follow as soon as we know that these two representations are equivalent. To see this it is enough to verify the trace identity

$$\mathrm{Tr}(T_n) - 2\,\mathrm{Tr}(T'_n) = \mathrm{Tr}(T_{k-2}(n)) \tag{23}$$

for all $(n, N) = 1$. We check these identities by considering the different contributions to the traces of T_n and T'_n in Theorem 4.42.

The contribution of the scalar term yields

$$\begin{aligned}\mathrm{Tr}(T_n)_{scalar} - 2\,\mathrm{Tr}(T'_n)_{scalar} &= \delta_{n=\square} n^{\frac{k}{2}-1}\frac{k-1}{12}\left[N\psi(N) - 2\frac{N}{p}\cdot\psi(\frac{N}{p})\right] \\ &= \delta_{n=\square} n^{\frac{k}{2}-1}\frac{k-1}{12}\frac{N}{p}\cdot\psi(\frac{N}{p})\,[p\psi(p)-2] = \delta_{n=\square} n^{\frac{k-2}{2}}\frac{k-1}{12}\frac{N}{p}\cdot\psi(\frac{N}{p})\varphi(p).\end{aligned}$$

This matches exactly the main term of $\mathrm{Tr}(T_{k-2}(n))$ given by Theorem 5.79.

The parabolic contribution yields

$$\mathrm{Tr}(T_n)_{parabolic} - 2\,\mathrm{Tr}(T'_n)_{parabolic} = \delta_{n=\square} n^{\frac{k-1}{2}}[2^{w(N)-1} - 2^{w(\frac{N}{p})}] = 0.$$

Note that $w(N) - 1 = w(\frac{N}{p})$ holds only if $p^2 \nmid N$.

The hyperbolic contribution is given by

$$\begin{aligned}
&\operatorname{Tr}(T_n)_{hyperbolic} - 2\operatorname{Tr}(T_n')_{hyperbolic} \\
&\quad = -\frac{1}{2} \sum_{\substack{t\in\mathbb{Z},\\ t^2-4n=\square}} \frac{\min(|\lambda_n^+(t)|,|\lambda_n^-(t)|)^{k-1}}{|\lambda_n^+(t)-\lambda_n^-(t)|} \sum_{f^2|t^2-4n} \phi(f)[2^{\omega(N)} - 2\cdot 2^{\omega(\frac{N}{p})}] = 0.
\end{aligned}$$

Finally we compute the elliptic contribution:

$$\begin{aligned}
&\operatorname{Tr}(T_n)_{elliptic} - 2\operatorname{Tr}(T_n')_{elliptic} \\
&= -\frac{1}{2} \sum_{\substack{t\in\mathbb{Z},\\ t^2-4n<0}} \frac{\lambda_n^+(t)^{k-1}-\lambda_n^-(t)^{k-1}}{\lambda_n^+(t)-\lambda_n^-(t)} \sum_{f^2|t^2-4n} \frac{h((t^2-4n)/f^2)}{\sharp(S^\times_{(t^2-4n)/f^2}/\{\pm\})} \\
&\qquad \cdot \prod_{p'|\frac{N}{p}} \left(1+\left\{\frac{(t^2-4n)/f^2}{p'}\right\}\right)\cdot\left[\left(1+\left\{\frac{(t^2-4n)/f^2}{p}\right\}\right)-2\right] \\
&= \frac{1}{2} \sum_{\substack{t\in\mathbb{Z},\\ t^2-4n<0}} \frac{\lambda_n^+(t)^{k-1}-\lambda_n^-(t)^{k-1}}{\lambda_n^+(t)-\lambda_n^-(t)} \sum_{f^2|t^2-4n} \frac{h((t^2-4n)/f^2)}{\sharp(S^\times_{(t^2-4n)/f^2}/\{\pm\})} \\
&\qquad \cdot \prod_{p'|\frac{N}{p}} \left(1+\left\{\frac{(t^2-4n)/f^2}{p'}\right\}\right)\cdot\left(1-\left\{\frac{(t^2-4n)/f^2}{p}\right\}\right).
\end{aligned}$$

According to Theorem 5.79 this agrees with

$$\operatorname{Tr}(T_{k-2}(n)) - \delta_{n=\square} n^{\frac{k-2}{2}} \frac{k-1}{12}\cdot\frac{N}{p}\psi(\frac{N}{p})\varphi(p).$$

Thus the trace equality (23) is established and the proof is complete. □

Remark 6.6. Note that using the appropriate trace formula for $(n,N),(n,\operatorname{disc}(O))\neq 1$ one can even show the trace identity

$$\operatorname{Tr}(T_n) = \operatorname{Tr}(T_{k-2}(n)) + \operatorname{Tr}(T_n|_{S_k(\frac{N}{p},\mathrm{Id})\oplus S_k(\frac{N}{p},\mathrm{Id})})$$

for all n.

Exercise 16. After doing Exercise 15 modify the statement of Theorem 6.2 accordingly and proof it. One can use the Eichler's version of the trace formula for $k=2$ stated in Remark 4.43.

References

1. Heinrich Brandt, *Zur Zahlentheorie der Quaternionen.*, Jahresbericht der Deutschen Mathematiker-Vereinigung **53** (1943), 23–57.
2. Martin Eichler, *Über die Darstellbarkeit von Modulformen durch Thetareihen*, J. reine angew. Math **195** (1955), 156–171.

3. ______, *The basis problem for modular forms and the traces of the Hecke operators*, Modular functions of one variable I, Springer, 1973, pp. 75–152.
4. Erich Hecke, *Analytische Arithmetik der positiven quadratischen Formen*, Munksgaard, 1940.
5. Hiroaki Hijikata, Arnold K Pizer, and Thomas R Shemanske, *The basis problem for modular forms on $\gamma_0(n)$*, vol. 418, American Mathematical Soc., 1989.
6. Henryk Iwaniec, *Topics in classical automorphic forms*, vol. 17, American Mathematical Soc., 1997.
7. Steven George Krantz, *Function theory of several complex variables*, vol. 340, American Mathematical Soc., 2001.
8. Toshitsune Miyake, *Modular forms*, Springer Science & Business Media, 2006.
9. John Voight, *Quaternion algebras*, Version v0 **9** (2018).

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