Introduction to Mathematical Hydrodynamics

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Abstract

These few pages are (yet incomplete) notes for a lecture series given in the University of Bonn in the Winter Semester 2023-2024. As such, some parts are still missing, and mistakes are likely to form a dense subset of the text. Please feel free to send me any comments, suggestions, or found mistakes.

The aim, throughout these pages, is to provide the student with an elementary and self-contained introduction to the modern mathematical study of fluid dynamics. No knowledge of physics or PDEs is required to understand any of the explanations, although mastery of (reasonably) basic functional analysis is assumed.

However, despite the introductory level of these notes, we should point out that they cover some topics which are currently the focus of very active research, and some of the results presented (and proved) are relatively recent.

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Chapter 1

Introduction to Non-Linear PDEs

We start these notes by presenting the main themes and specific issues of non-linear PDEs by an example that is particularly relevant in hydrodynamics: Burgers equation. We will start by shortly explaining the model before exploring the issues that are linked with it's resolution.

1.1 PDEs and ODEs: What Differences?

It is hardly necessary to justify the importance of differential equations: these are at the heart of classical physics, and are involved in numerous non-physical phenomena, such as economics, population dynamics, genetics, *etc.* These equations link the time evolution of a quantity to its value through a relation of the form

$$y' = f(y), \qquad \text{where } f : X \longrightarrow X \text{ and } X \text{ is Banach.} \tag{1.1}$$

and a fairly complete theory of these equations has been in existence for the better part of two centuries. They are sometimes called Ordinary Differential Equations, or ODEs, in reference to the fact that they do not involve partial derivatives. Does that mean that Partial Differential Equations, or PDEs, should be considered *extraordinary*? It is up to the reader to decide, and it is the purpose of this chapter to help him understand what are the essential differences between ODEs and PDEs. Before we start, let us list and comment some of the main features of ODEs.

Cauchy-Lipschitz theory. Under very general conditions, any ODE (1.1) equipped with an initial value $y(0) = y_0$ has a unique maximal solution $y :]T_-, T_+[\longrightarrow X$. This is a *general theory* of ODEs.

Non-linear ODEs. When the ODE is non-linear, solutions may be non-global: $-\infty < T_-$ and/or $T_+ < +\infty$. For example, in the ODE $y' = y^2$, the solution is given by

$$y(t) = \frac{1}{\frac{1}{y_0} - t},$$

and exists up to time $T = 1/y_0$.

Blow-up. If $T_\pm$ is finite, then the solution must eventually exit any compact subset of X in its lifespan. When X has finite dimension, this means that

$$\left\|y(t)\right\|_X \longrightarrow +\infty \qquad \text{as } t \to T_\pm.$$

What about PDEs? Firstly, it should be said that *there is no general theory* of PDEs. Very few theorems are "universal"[1], in that most results or methods apply to a specific class of PDEs.

[1] The Cauchy-Kowaleski theorem is a notable exception.

On the one hand, this means that the analysis of these equations is an extremely vast field. And secondly, it means that it is also very much example driven: intuition is built on the knowledge of many examples.

Consequently, in the rest of this chapter, we will present a specific PDE, Burgers equation, which has the advantage of allowing many explicit computations, while providing precious insight on the behavior of non-linear PDEs.

1.2 Informal Derivation of Burgers Equation

We consider the motion of a set of particles on the real line $\mathbb{R}$, which we will describe continuously by a density function $u(t,x) \geq 0$. In that way, the number of particles $N(t)$ in an interval $I \subset \mathbb{R}$ at a time $t \in \mathbb{R}$ is exactly

$$N(t) = \int_I u(t,x)\,\mathrm{d}x.$$

Now, we make the assumption that the particles move to the right (in the direction of increasing $x \in \mathbb{R}$) according to some kind of self-encouragement law: the more particles there are at a given place, the faster will their motion be. More precisely, we assume that the average velocity $v(t,x)$ of particles at a point $x \in \mathbb{R}$ and a time $t \in \mathbb{R}$ is given by the law

$$v(t,x) = \frac{1}{2}u(t,x).$$

With these assumptions, we wonder whether we will be able to derive a PDE law for $u(t,x)$. In most cases in mechanics, the derivation of a differential equation law is the result of a conservation equation. Here, we will attempt to count the number of particles $N(t)$ that are in a given interval $I = [a,b] \subset \mathbb{R}$. This number evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_I u(t,x)\,\mathrm{d}x = \int_I \partial_t u(t,x)\,\mathrm{d}x.$$

On the other hand, change of the total number $N(t)$ is due to particles entering or leaving by the edges, a or b. For example, at a given time, the flux $\phi_a(t)$ of particles entering I through a will be

$$\phi_a(t) = v(t,a) \times u(t,a) = \frac{1}{2}u(t,a)^2,$$

because ϕ_a is the product of the number of particles at $x = a$ by the speed of these particles. Similarly, the flux $\phi_b(t)$ of particles entering by b is $\phi_b(t) = -\frac{1}{2}u(t,b)^2$, where the opposite sign accounts for the fact that $v(t,b)$ is the speed of particles *leaving* the interval. Therefore, the total number of particles in I evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \phi_a(t) + \phi_b(t),$$

or in other words

$$\int_I \partial_t u(t,x)\,\mathrm{d}x + \frac{1}{2}\Big(u(t,b)^2 - u(t,a)^2\Big) = 0.$$

By recognizing an integral in the difference above, we find that

$$\int_I \Big(\partial_t u(t,x) + \frac{1}{2}\partial_x\left(u(t,x)^2\right)\Big)\,\mathrm{d}x = 0,$$

and since that must be true at all times and for all intervals $I \subset \mathbb{R}$, we deduce the *Burgers equation*:

$$\partial_t u + \frac{1}{2}\partial_x(u^2) = 0. \tag{1.2}$$

Remark 1. As is usual in the Analysis of PDEs, we have omitted the dependencies on (t, x) when writing (1.2), and will continue to do so in the rest of these lecture notes. The reason for this is twofold. First and foremost, it is a matter of mercy for the reader (and the author) to not have to repeat needlessly and endlessly the arguments (t, x), except perhaps in cases rare where it provides some clarification. Secondly, and this is equally important, we will often work with L^p spaces or spaces of distributions. Therefore, the notation $u = u(t, x)$ may refer to a function, a class of function defined up to a measure zero set, or even a bounded linear form defined on a Banach space of functions. Or all three at the same time! And there usually is no ambiguity in doing so...

Let us introduce one last notation: when $f(t, x)$ is a function of both space and time, we will note $f(t)$ the restriction $x \mapsto f(t, x)$ at a fixed $t \in \mathbb{R}$, whenever it makes sense to do so.

Remark 2. Burgers equation (1.2) is a toy model for ideal incompressible fluids. This means that it is a simple equation that mirrors the behavior of more complicated ones that are actually used in fluid dynamics. In particular, if u is rather interpreted as a (vector) momentum density, and if we take into account pressure forces, then we may obtain the Euler equations, which play a key part in this course.

Remark 3. Also note that the self-driving assumption is at the basis of the non-linear character of the equation: because u is both the particle density and the fluid velocity, the rightwards flux at a point a is a non-linear function of the solution $\phi_a = u(a)^2$. If the particle velocity was a fixed function, say $V(t, x)$, then the flux would have been $\phi(a) = V(a)u(a)$, thus giving a linear PDE. Generally speaking, this kind of self-driving, or self-reference, is the source of non-linear behavior.

1.3 Solving Burgers Equation: Classical Solutions

In the previous paragraph, we have derived an evolution equation for $u(t, x)$. An evolution equation is the PDE equivalent of a differential equation: it describes the time dynamics of a phenomenon with respect to its space dynamics. In good conditions, evolution equations can be used to compute solutions from a given initial datum. For example, imagine that the solution u of (1.2) is known at time t. Then, in a perfect world, we should have (in physicist language)

$$u(t + \mathrm{d}t) = u(t) + \frac{1}{2}\partial_x\big(u(t)^2\big)\,\mathrm{d}t.$$

Of course, things are never so simple. As an illustration of what can go wrong, imagine that $u(t)$ is a C^1 function (and not C^2). Then, from the naive equation above, we would expect that $u(t + \mathrm{d}t)$ is C^0 and not C^1, so that the solution instantaneously loses regularity! While this is a rather pessimistic evaluation of the situation in the case of Burgers (we will solve the equation below), this is actually a real problem: unlike ODEs, it is not at all obvious that a PDE has a sensible solution stemming from any given initial datum (even on a short time interval). A good illustration of this fact is the reverse heat equation

$$\partial_t u + \Delta u = 0,$$

whose solutions are given by Fourier transform (in the space variable only)

$$\widehat{u}(t, \xi) = e^{t|\xi|^2}\widehat{u}(0, \xi).$$

The formula above shows that if $u(0)$ is not C^∞, then the solution at any time $t > 0$ cannot be even a locally integrable function. In fact, because of the growing exponential in the equation above, it is not obvious at all that $u(t)$ can even be defined as a tempered distribution!

The first step in the study of a PDE is therefore to make sure that solutions exist and are unique for a given initial datum. As in the case of ODEs, this is called the Cauchy problem, or the initial value problem. Let's look at this for Burgers equation: for a given bounded $u_0 \in C^\infty$, we want to find a u such that

$$\begin{cases} \partial_t u + \frac{1}{2}\partial_x(u^2) = 0 \\ u(0,x) = u_0(x). \end{cases} \tag{1.3}$$

The key idea is the notion of characteristic: a characteristic is a curve $y(t)$ such that the quantity $u(t, y(t))$ is independent of time. Consider a smooth function $y(t)$, a smooth solution u, and define $g(t) := u(t, y(t))$. Then,

$$g'(t) = \partial_t u(t, y(t)) + y'(t)\partial_x u(t, y(t)),$$

so that y is a characteristic as soon as it solves the differential equation $y' = u(t, y)$, in which case we have $g(t) = g(0) = u_0(y(0))$. We therefore consider y to be a solution of the ODE problem

$$\begin{cases} y'(t) = u\big(t, y(t)\big) \\ y(0) = x, \end{cases}$$

so that $u(t, y(t)) = u_0(x)$ for any $x \in \mathbb{R}$. Because $y'(t) = g(t) = u_0(x)$, we may express $y(t)$ as a function of the data: we have $y(t) = \phi_t(x)$ where

$$\phi_t(x) = x + tu_0(x),$$

and expressing $u(t, x)$ as a function of (t, x) is a matter of inverting ϕ_t. We have an expression of the solution as

$$u(t, x) = u_0(\phi_t^{-1}(x)). \tag{1.4}$$

Of course, for all this to work, ϕ_t must be invertible. Recall that a C^1 function $\mathbb{R} \longrightarrow \mathbb{R}$ is invertible with C^1 continuous inverse if and only if it is (strictly) monotonous. Therefore, ϕ_t will be invertible with a C^1 inverse if and only if the derivative $\partial_x \phi_t$ never cancels. Because

$$\partial_x \phi_t(x) = 1 + t\partial_x u_0(x),$$

this will always be true for small enough times, provided that the derivative $\partial_x u_0$ is bounded. However, unless u_0 is nondecreasing, this happens first at time (consider only nonnegative times)

$$T^* = \frac{1}{-\min \partial_x u_0}.$$

Therefore, just as in the case of non-linear ODEs, we have shown that the Cauchy problem (1.3) can be uniquely solved by a smooth solution, but only on a given time interval. At time T^*, the solution ceases to be smooth (or even continuous), and our analysis breaks down. This type of result is known as a *local well-posedness* result (as opposed to the case when solutions are global). As we have explained, proving (local) well-posedness is already non-trivial in the case of most PDEs.

The situation of Burgers equation is quite similar to what we may already see in the case of non-linear ODEs. For example, the differential equation $y' = y^2$ with initial datum $y_0 \neq 0$ has solution

$$y(t) = \frac{1}{\frac{1}{y_0} - t},$$

which blows-up at time $T^* = 1/y_0$. In both cases, non-linear equations may produce finite time blow-up, and the lifespan of solutions depends on how large the initial datum is.

1.4 Introducing Weak Solutions

In the case of non-linear ODEs, finite time-blow-up is characterized by the norm of the solution becoming infinite. While this seems to make sense in our example, we also see profound differences: if $T^* < +\infty$, then the solution (1.4) satisfies

$$\|u(t)\|_{C^1} \longrightarrow +\infty \qquad \text{as } t \to T^*,$$

but the form (1.4) of the solution also shows that it remains bounded at all times

$$\|u(t)\|_{L^\infty} = \|u_0\|_{L^\infty}.$$

This remark makes us of course wonder whether it is possible to extend the smooth solution u beyond T^* into a solution that will remain bounded, although it cannot be continuous. In order to make sense of this, the derivatives in the PDE must be understood in the sense of distributions. Let us explain.

If $u : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is a (smooth) solution of the Cauchy problem (1.3), then for any compactly supported function $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$, we have[2]

$$\int_{\mathbb{R}_+} \int \Big(\partial_t u + \frac{1}{2} \partial_x (u^2) \Big) \phi \, \mathrm{d}x \, \mathrm{d}t = 0.$$

By integrating by parts, we see that this is equivalent to

$$\int u_0(x) \phi(0, x) \, \mathrm{d}x + \int_{\mathbb{R}_+} \int \Big(u \partial_t \phi + + \frac{1}{2} u^2 \partial_x \phi \Big) \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{1.5}$$

The reverse is equally true: if (1.5) holds for all compactly supported $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$, then u must be a solution of Burgers equation with initial datum u_0. However, (1.5) has a notable advantage: it makes sense for any bounded function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$.

We therefore say that a bounded function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is a *weak solution* of Burgers equation with initial datum $u_0 \in L^\infty(\mathbb{R})$ if (1.5) holds for all compactly supported $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$.

Remark 4. In accordance with the french training of the author, we will use the Schwarz notation $\mathcal{D}(\Omega)$ for the space of C^∞ and compactly supported functions on an open set $\Omega \subset \mathbb{R}^n$, for $n \geq 1$. The reader may find other notation in the literature, such as $C_c^\infty(\Omega)$. Consequently, we note $\mathcal{D}'(\Omega)$ the space of distributions on Ω, which is the (topological) dual of $\mathcal{D}(\Omega)$. This space is also sometimes noted $C^{-\infty}(\Omega)$, although rarely in works relating to the analysis of PDEs.

Remark 5. In order for the integral in (1.5) to make sense, it is crucial that the product $u \partial_x u$ be expressed as a total derivative $\frac{1}{2} \partial_x (u^2)$. This reflects the fact that $\frac{1}{2} \partial_x (u^2)$ makes sense as a distribution (it is the derivative of a L^∞ function), but $u \partial_x u$ does not, as the product of two distributions is not well-defined.

Remark 6. We make one last remark: there is a slight difference between the weak form of Burgers equation (1.5) and what we would have by simply using distribution theory: this is due to the presence of the initial datum, which forces us to take test functions[3] in $C^\infty(\mathbb{R}_+ \times \mathbb{R})$ instead

[2] When a domain is not given, integrals should be understood to bear on $\mathbb{R}$, or on $\mathbb{R}^d$ when we work in several space dimensions.

[3] Test functions are the ϕ functions that are involved in the weak form of a PDE, or a distributional equality. In general, if $\Omega \subset \mathbb{R}^d$ is a product of open and closed sets, we note $\mathcal{D}(\Omega)$ the space of C^∞ functions that are supported in compact subsets of Ω.

of test functions defined on an open[4] set $\mathcal{D}(\mathbb{R}_+^* \times \mathbb{R})$. This problem can be ignored by extending u to a all times, setting $u(t) = 0$ for $t < 0$, and noting that (1.5) is equivalent to

$$\partial_t u + \frac{1}{2}\partial_x(u^2) = \delta_0(t) \otimes u_0(x) \qquad \text{in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}),$$

where $\delta_0(t)$ is the Dirac delta in time and the tensor product $\delta_0(t) \otimes u_0(x)$ is defined by

$$\langle \delta_0(t) \otimes u_0(x), \phi(t,x) \rangle = \int \phi(0,x) u_0(x) \,\mathrm{d}x,$$

for all $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R})$.

Now that we have a notion of solution that makes sense for bounded functions, the question is whether weak solutions exist and are globally defined (that is for all times). Just like local well-posedness, the construction of global weak solutions is also quite a challenge for most PDEs. While proving that there is a global weak solution of Burgers equation for any bounded $u_0 \in L^\infty$ goes far beyond the scope of this introduction chapter, we may provide a simple example.

We consider the propagation of a shock front in Burgers equation: assume that $u(t,x)$ is given by

$$u(t,x) = \begin{cases} a & \text{if } x < f(t), \\ b & \text{if } x \geq f(t), \end{cases}$$

where real $a \neq b$ and the singularity propagates at a given speed, so $f(t) = ct$ with $c \in \mathbb{R}$. Let us find out on what condition u is a solution of Burgers equation. Consider a test function $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R})$ and compute

$$I_1 + I_2 := \iint u\partial\phi \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\iint u^2 \partial_x \phi \,\mathrm{d}x\,\mathrm{d}t.$$

The second integral I_2 has value

$$\begin{aligned} I_2 &= \frac{1}{2}a^2 \int \int_{-\infty}^{ct} \partial_x \phi \,\mathrm{d}x\,\mathrm{d}t + \int \int_{ct}^{+\infty} \partial_x \phi \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}a^2 \int \phi(t,ct)\,\mathrm{d}t - \frac{1}{2}b^2 \int \phi(t,ct)\,\mathrm{d}t \\ &= \frac{1}{2}(a^2 - b^2) \int \phi(t,ct)\,\mathrm{d}t. \end{aligned}$$

On the other hand, the integral I_1 can be computed similarly: by Fubini's theorem,

$$\begin{aligned} I_1 &= a \iint \mathbb{1}_{x<ct} \partial_t \phi(t,x)\,\mathrm{d}x\,\mathrm{d}t + b \iint \mathbb{1}_{x>ct} \partial_t \phi(t,x))\,\mathrm{d}x\,\mathrm{d}t \\ &= \int \int_{x/c}^{+\infty} \partial_t \phi(t,x)\,\mathrm{d}t\,\mathrm{d}x + b \int_{-\infty}^{x/c} \partial_t \phi(t,x)\,\mathrm{d}t\,\mathrm{d}x \\ &= (b-a) \int \phi(x/c, x)\,\mathrm{d}x = c(b-a) \int \phi(x, cx)\,\mathrm{d}x. \end{aligned}$$

Therefore, we see that u is a weak solution of the Burgers equation if and only if the shock propagates at a speed of

$$c = \frac{1}{2}\frac{a^2 - b^2}{a - b} = \frac{1}{2}(a+b). \tag{1.6}$$

This condition is a particular case of the *Rankine-Hugoniot condition*, which bear on jump discontinuities in conservation laws, of which Burgers equation is an example. Note that, as expected, the weak solution u we have constructed has a L^∞ norm that is constant in time.

[4] We note $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$.

Remark 7. Of course, the computations above may be understood (and directly done) in the language of distributions:

$$\partial_t u + \frac{1}{2}\partial_x(u^2) = c(a-b)\delta_{f(t)} + \frac{1}{2}(b^2-a^2)\delta_{f(t)}.$$

In the above, $\delta_z(x) = \delta_0(x-z)$ notes the Dirac delta distribution at a point $z \in \mathbb{R}$.

Remark 8. When a solution is sufficiently regular so that all the derivatives in a PDE can be computed explicitly, it is called a *classical solution*. For example, if a weak solution of the Burgers equation u lies in $C^1(\mathbb{R}_+ \times \mathbb{R})$, then it is in fact a classical solution. A *strong solution* is usually the product of a local (or global) well-posedness result, although the terminology varies from author to author.

1.5 Uniqueness Issues and the Well-Posedness Problem

Now that we have explained the notion of weak solutions, which may extend a classical solution beyond the finite time of blow-up, we present its main flaw: while weak solutions may globally solve a Cauchy problem, they are in general not unique.

We give an example based on propagating shock fronts. Consider the bounded function given

$$u_a(t,x) = \begin{cases} a & \text{if } 0 < x < \frac{1}{2}at \\ -a & \text{if } -\frac{1}{2}at < x < 0 \\ 0 & \text{elswhere,} \end{cases} \qquad \text{for } a > 0.$$

Then all the discontinuities in u propagate at speeds that fulfills the Rankine-Hugoniot condition (1.6), so that u is a weak solution of Burgers equation, and is associated to the initial datum $u_0 = 0$. Of course, $u = 0$ is also a weak solution of the Cauchy problem... Here, we have an example of an infinity of different weak solutions that all agree at time $t = 0$. The lesson is that weak is usually too weak to get uniqueness.

Of course, $u = 0$ is the only *classical solution* of Burgers equation with initial datum $u_0 = 0$. But the problem is that, for generic initial data, the unique classical solution only lasts for so long. Once it becomes discontinuous (it blows-up), it can be extended into a weak solution, but this extension is usually not unique: there may be infinitely many different extensions.

In general, the type of questions we will ask when studying a PDE are the following:

Local Well-Posedness: can we prove existence and uniqueness of (strong) solutions on a finite time interval $[0, T^*[$? How does the lifespan of strong solutions depend on the initial data? Can we have $T^* = +\infty$?

Blow-Up: is T^* finite? Is it possible to construct blow-up solutions? What happens at the time of blow-up? Which norms tend to infinity?

Global Weak Solutions: do global weak solutions exist? Can they be unique?

Qualitative Properties: given that solutions are rarely explicit, is it possible to determine their qualitative behavior?

Remark 9. Burgers equation is a part of a more general class of problems, scalar conservation laws, which have been intensively studied. In fact, there is a way to select a weak solution (called the entropy solution) in order to obtain a uniqueness theorem for L^∞ solutions. See Chapter 11 of [15] (pp. 609-658) for an introduction to these problems.

Chapter 2

Incompressible Euler Equations

In this chapter, we introduce one of the most basic models in fluid dynamics, the incompressible Euler equations. Once we have shortly presented the underlying physics of the system, we will turn to our main goal: showing the existence and uniqueness of solutions for the Cauchy problem.

2.1 Introduction: Derivation of the System

Consider a fluid occupying the whole Euclidean space $\mathbb{R}^d$, with $d \geq 2$. We make the following physical assumptions:

- the fluid is governed by Newtonian mechanics,
- the fluid is ideal, and therefore has no viscosity (it is *inviscid*),
- the fluid has a constant density.

All of the above are very reasonable hypotheses to make when studying, say, the flow of water on short enough times.[1] Just as before, we will seek to write a PDE model that describes the dynamics of the fluid. Let us introduce the following notation:

- $u(t,x) \in \mathbb{R}^d$ is the (vector) velocity of the fluid particles at a point $x \in \mathbb{R}^d$ and a time $t \in \mathbb{R}$. We work in $d \geq 2$ dimensions of space;[2]
- $\pi(t,x) \in \mathbb{R}$ is the scalar pressure field in the fluid; $\rho > 0$ is the constant density (and does not depend on time or space).

Just as in the case of Burgers equation, we resort to conservation laws to derive the fluid equations. As this is a modelling step, our computations are formal. Let $\Omega \subset \mathbb{R}^d$ be an fixed arbitrary domain. We start by writing a conservation of mass equation: the variation of mass inside Ω is due to the flux of mass through the boundary $\partial\Omega$, in other words

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho \, \mathrm{d}x = - \oint_{\partial\Omega} \rho u \cdot n \mathrm{d}\sigma,$$

where n is the outward normal unit vector on the boundary $\partial\Omega$ and σ is the Euclidean surface measure. Integrating by parts provides

$$\int_\Omega \Big(\partial_t \rho + \mathrm{div}(\rho u) \Big) \, \mathrm{d}x = 0,$$

[1] We should also add that the inviscid assumption is accurate for flows that are far enough from an interface, such as the edge of a container.

[2] In practice, $d = 2$ or $d = 3$, but keeping a general value for d helps to understand the role of the dimension in the proofs. Most of the time, d appears in Sobolev embeddings.

and since ρ is assumed to be constant, we get the conservation of mass equation:

$$\operatorname{div}(u) = 0. \tag{2.1}$$

Remark 10. It may be puzzling at first that we only consider fluids on at least two dimensions of space $d \geq 2$. Would a fluid in a thin pipe not be modeled by a one-dimensional equation? However, the mass equation (2.1) above explains why: in one dimension it reduces to $\partial_x u = 0$, so that u must be a function of time only. One dimension fluids with constant mass are not very interesting... Of course, the situation is very different for compressible fluids, but that is a very different topic!

Now, we write a conservation of momentum equation, that is Newton's second law applied to the volume Ω. However, Ω is not an isolated mechanical system, so momentum flows in and out of Ω as the fluid enters and leaves Ω, and we have to take this into account in addition to the pressure forces applied on Ω. We have:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho u \,\mathrm{d}x}_{\text{variation of momentum}} = \underbrace{-\oint_{\partial\Omega} \pi n \mathrm{d}\sigma}_{\text{pressure forces}} \underbrace{- \oint_{\partial\Omega} \rho u(u\cdot n)\mathrm{d}\sigma}_{\text{momentum flux}}.$$

We integrate by parts inside the surface integrals. For example, the momentum flux integral reads

$$-\oint_{\partial\Omega} \rho u(u\cdot n)\mathrm{d}\sigma = -\sum_j \oint_{\partial\Omega} \rho u_j e_j (u\cdot n)\mathrm{d}\sigma = \sum_j \int_\Omega \operatorname{div}(\rho u_j u) e_j \,\mathrm{d}x,$$

and the same can be done inside the pressure integral. In the above, (e_j) is the canonical basis of $\mathbb{R}^d$. Before writing the momentum equation, let us introduce a convenient notation: if $A_{kj}(t,x)$ is a field of $d\times d$ matrices, we define the divergence of A to be the vector field $\operatorname{div}(A) = \sum_k \partial_k(A_{kj})e_j$. By using this notation, we have:

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla\pi = 0. \tag{2.2}$$

In the equation above, the tensor product $u \otimes u$ denotes the matrix whose coefficients are given by $[u\otimes u]_{ij} = u_j u_j$.

Equations (2.1) and (2.2) form a system of $d+1$ equations for the $d+1$ unknowns $(u_1, ..., u_d, \pi)$, and are called the *Euler system*, or the *Euler equations*. Most of the time, we will ignore the constant density, which does not affect any of the computations. This is equivalent to working in a set of units where $\rho = 1$, so that the Euler equations read

$$\begin{cases} \partial_t u + \operatorname{div}(u\otimes u) + \nabla\pi = 0 \\ \operatorname{div}(u) = 0. \end{cases} \tag{2.3}$$

Before moving on, we must notice the proximity of the Euler system with Burgers equation. By using equation (2.1), we may rewrite the divergence in (2.3) as

$$\begin{aligned} \operatorname{div}(u\otimes u) &= \sum_{k,j} \partial_k(u_k u_j)e_j = \sum_{k,j}\Big(\partial_k u_k u_j e_j + u_k \partial_k u_j e_j\Big) \\ &= \operatorname{div}(u)u + (u\cdot\nabla)u = (u\cdot\nabla)u, \end{aligned}$$

and the Euler system becomes

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla\pi = 0 \\ \operatorname{div}(u) = 0. \end{cases} \tag{2.4}$$

Imagine for an instant that $u(t,x) = u_1(t,x_1)e_1$ is a solution that depends only on the first space variable and colinear to e_1. Then the momentum equation above simplifies and we obtain instead

$$\partial_t u_1 + u_1\partial_1 u_1 + \partial_1 \pi = 0,$$

which is the Burgers equation with an added pressure term. Of course, this situation can never really happen: the divergence equation (2.1) would force u to be constant. But the proximity of the two PDEs is nevertheless striking, while the conservation of mass implies that there should also be profound differences. As we will see, this is the case.

Remark 11. We must make one remark on the operator $(u \cdot \nabla)$ introduced above. While the notation may seem artificial at first[3], it is not only convenient but carries deep meaning. For example, if $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a vector field with

$$\partial_t f + (u \cdot \nabla) f = 0,$$

then we see that all the components of f solve the same transport equation:

$$\partial_t f_j + u \cdot \nabla f_j = 0.$$

In particular, the function f is constant along all the characteristics associated to the vector field u. If $\phi_t(x)$ is the flow of u defined by $\partial_t \phi_t = u(t,\phi_t)$ with $\phi_0 = \mathrm{Id}$, then computations identical to those in the previous chapter show that $f(t,\phi_t(x)) = f(0,x)$. Although we will not use this fact in the current chapter, it will be absolutely crucial for a more advanced investigation of the Euler equations later on.

2.2 Computing the Pressure and Leray Projection

We continue to explore the Euler equations at a formal level with a few remarks about the pressure function. As we have explained above, the pressure π is, just as the velocity u, an unknown of system (2.4) in its own right, and must be determined in any resolution of the equations. However, as the astute reader may have noticed, it is not a *dynamic* unknown: while there is an evolution equation for u (an equation giving $\partial_t u$ as a function of u and its derivatives), it is not the case for π. The practical consequence of this is that, at any time $t \in \mathbb{R}$, the pressure $\pi(t)$ can be expressed as a function of $u(t)$. This is the goal of this section.

The starting point is that the velocity field u is divergence-free $\mathrm{div}(u) = 0$. It is therefore tempting to take the divergence of the momentum equation:

$$\mathrm{div}(\partial_t u) + \mathrm{div}\big((u\cdot\nabla)u\big) + \mathrm{div}(\nabla\pi) = 0.$$

On the one hand, $\mathrm{div}(\partial_t u) = 0$ by the Schwarz theorem, while $\mathrm{div}(\nabla\pi) = \Delta\pi$. On the other hand, the remaining term is

$$\mathrm{div}\big((u\cdot\nabla)u\big) = \sum_{k,j} \partial_j\big(u_k\partial_k u_j\big)$$

What we have left is an elliptic equation whose solution is the pressure related to a quantity that depends only on u, namely

$$-\Delta\pi = \sum_{j,k} \partial_j\big(u_k\partial_k u_j\big)$$

[3]Just as many notations do in vector calculus. As much as possible, we will refrain from using the “nabla” operator ∇ in a too confusing way, and avoid as much as possible the notations $\nabla\times$ (for the curl), $\nabla\cdot$ (for the divergence), ∇^2 (for the Laplacian), *etc.*

Before solving the equation above, we attract the reader's attention to the fact that sums like the one above will be omnipresent in our computations, and will therefore weigh on the clarity of the text. For that reason, it is usual to omit the summation symbol, making it implicit whenever there is a repeated index (this practice is due to Einstein, and bears his name[4]). For example, in the sum above, the indices $j, k = 1, ..., d$ are repeated, to the elliptic equation above can be abridged as

$$-\Delta\pi = \partial_j\big(u_k\partial_k u_j\big). \tag{2.5}$$

There are many equivalent ways to solve this elliptic equation. One of them is to use an elementary solution of the Laplace operator: a function $E(x)$ whose Laplacian is a Dirac delta $-\Delta E = \delta_0$, and we will use this approach later on. For the moment, we solve equation (2.5) by a Fourier transform method. If $f(t, x)$ is a smooth integrable function, we note $\widehat{f}(t, \xi)$ the Fourier transform with respect to space:

$$\widehat{f}(t,\xi) = \int f(t,x)e^{-ix\cdot\xi}\,\mathrm{d}x.$$

The Fourier transform is a linear operation with respect to f. This prompts us to consider it as an operator $\mathcal{F}$, so that $\widehat{f}(t,\xi) = \mathcal{F}f(t,\xi)$. By taking the Fourier transform of both sides in (2.5), we obtain

$$\widehat{\pi}(t,\xi) = \frac{1}{|\xi|^2}\mathcal{F}\big[\partial_j(u_k\partial_k u_j)\big](t,\xi).$$

The pressure π can therefore be obtained by taking the reverse Fourier transform of the equation immediately above. It is in this way that we should understand the inversion of the Laplace operator: we have $(-\Delta)^{-1}f := \mathcal{F}^{-1}\big[|\xi|^{-2}\widehat{f}(\xi)\big]$. Therefore, the pressure force reads

$$\nabla\pi = \nabla(-\Delta)^{-1}\,\mathrm{div}\,\big((u\cdot\nabla)u\big).$$

By plugging this in the Euler system, we obtain an evolution equation which involves only the velocity field:

$$\partial_t u + \big(\mathrm{Id} + \nabla(-\Delta)^{-1}\,\mathrm{div}\,\big)\big((u\cdot\nabla)u\big) = 0. \tag{2.6}$$

Of course, all the computations in the ection have been entirely formal, but will be rigorously justified in the sequel. For now, we will be content with summarizing our motivational explanations in a definition.

Definition 12 (Leray projector)**.** *We define the Leray projection operator* $\mathbb{P} = \mathrm{Id} + \nabla(-\Delta)^{-1}\,\mathrm{div}$ *in the following way. For any vector field* $f \in L^2(\mathbb{R}^d;\mathbb{R}^d)$*, we define* $\mathbb{P}f$ *by its Fourier transform:*

$$\widehat{\mathbb{P}f}(\xi) := \widehat{f}(\xi) - \frac{\xi\xi_j}{|\xi|^2}\widehat{f}_j(\xi),$$

where, in the last summand, there is an implicit summation on the repeated index $j = 1, ..., d$*.*

Proposition 13. *The Leray projection is a bounded operator* $\mathbb{P} : L^2 \longrightarrow L^2$*.*

Proof. We have to prove that $\mathbb{P} : L^2 \longrightarrow L^2$ is a bounded operator. Consider $f \in L^2$. Note that the coefficients

$$\sigma_{jk}(\xi) = \delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}$$

are all bounded functions on $\mathbb{R}^d \setminus \{0\}$. Therefore, we can use Plancherel's identity to show that

$$\|\mathbb{P}f\|_{L^2} = C(d)\|\widehat{\mathbb{P}f}\|_{L^2} = C(d)\|\sigma.\widehat{f}\|_{L^2} \leq C(d)\|\sigma\|_{L^\infty}\|\widehat{f}\|_{L^2} = C(d)\|f\|_{L^2}.$$

$\square$

[4]Strictly speaking, the Einstein summation convention bears on repeated indices, one of which is covariant and the other which is contravariant. While this distinction is useful in Riemannian geometry, our Euclidean setting makes it completely superfluous: all indices will be noted as subscripts.

Remark 14. The operator $\mathbb{P}$ is a *Fourier multiplier*, an operator that is defined by multiplication of a function in the Fourier variable. The (matrix valued) multiplier function $\sigma(\xi)$ is called the *symbol* of $\mathbb{P}$. Fourier multipliers are a special class of pseudo-differential operators. Conversely, classical differential operators with constant coefficients are a special class of Fourier multipliers, those whose symbols are *polynomial* functions of ξ.

In general, Fourier multipliers are *non-local*. This means that the image under $\mathbb{P}$ of a (smooth) function supported in a given compact set $K \subset \mathbb{R}^d$ will not, in general, be supported inside that same set (or even have compact support). This is due to the fact that Fourier multipliers are in fact *convolution operators*. If T is a Fourier multiplier of symbol $m(\xi)$, then

$$Tf = \mathcal{F}^{-1}\big[m\widehat{f}\,\big] = \mathcal{F}^{-1}[m] * f.$$

It is possible to show that the only Fourier multipliers that do not exhibit a non-local behavior necessarily are differential operators with constant coefficients (see [14], paragraph 5.11.3).

The non-local behavior of the Euler equations is very strongly linked to the constant density assumption. As an illustration of this, imagine a fluid lying inside a finite tube with a mechanical pressure applied on one end, the other end being open. The fluid cannot move under the pressure unless some fluid immediately leaves the tube by the other end, otherwise a zone with a higher density would be created inside the tube! This thought experiment shows that acting on the fluid at one place has instantaneous consequences throughout the whole fluid. In other words, there is a non-local phenomenon. This effect is of course linked to the fact that the speed of sound is higher in fluids that have low compressibility, such as water: in a (mathematically ideal) incompressible fluid, the speed of sound is infinite.

Remark 15. The Leray projection operator is an explicit way of writing the Helmholtz decomposition of a vector field: any vector field $f \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ is the sum of a divergence-free vector field (in the sense of distributions) and the gradient of a function,

$$f = g + \nabla h \qquad \text{with } \operatorname{div}(g) = 0.$$

The field g is given by $\mathbb{P}f \in L^2$, while h is given by $\nabla h = -\nabla(-\Delta)^{-1}\operatorname{div}(f)$. Of course, the difficult part here is to show that h is a well defined object. We will not expand further here, referring the reader to [9], Lemma 1.1, Chapter 3. For the time being, note that $\mathbb{P}$ is the L^2-orthogonal projection on the closed subspace

$$L^2_\sigma := \Big\{u \in L^2(\mathbb{R}^d;\mathbb{R}^d), \quad \operatorname{div}(u) = 0\Big\}. \tag{2.7}$$

2.3 Energy Balance and Time-Space Norms

This paragraph revolves around the conservation of kinetic energy in an ideal fluid and its natural consequence, an adapted functional framework for the mathematical analysis of the Euler equations. As we have explained while deriving the equations, the only force acting on the fluid is the pressure force $-\nabla\pi$. It should therefore be possible to write an energy conservation law in the fluid. Here again, the computations stay at the formal level, as this is still a motivational paragraph.

Assume that u is a smooth solutions of the Euler equations that decays sufficiently fast at infinity, associated to the pressure function π. We start by considering the kinetic energy, which is distributed according to the density $e_c = \frac{1}{2}|u|^2$. The variation of the total kinetic energy in the fluid is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}|u|^2\,\mathrm{d}x = \int \partial_t u \cdot u\,\mathrm{d}x.$$

We may use the momentum equation to rewrite the time derivative. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}|u|^2\,\mathrm{d}x = -\int \nabla\pi\cdot u\,\mathrm{d}x - \int (u\cdot\nabla)u\cdot u\,\mathrm{d}x.$$

On the one hand, integration by parts shows that

$$-\int \nabla\pi\cdot u\,\mathrm{d}x = \int \pi\,\mathrm{div}(u)\,\mathrm{d}x = 0.$$

On the other hand, we may also perform an integration by parts in the second integral. We have:

$$\begin{aligned}
-\int (u\cdot\nabla)u\cdot u\,\mathrm{d}x &= -\int u_k\partial_k u_j u_j\,\mathrm{d}x \\
&= \int u_j\partial_k(u_k u_j)\,\mathrm{d}x \\
&= \int u_j u_k\partial_k u_j\,\mathrm{d}x = \int (u\cdot\nabla)u\cdot u\,\mathrm{d}x.
\end{aligned} \tag{2.8}$$

Because this integral is equal to its opposite, it must be zero. We deduce that the total kinetic energy is independent of time. In other words,

$$\forall t\in\mathbb{R},\qquad \int |u(t,x)|^2\,\mathrm{d}x = \int |u(0,x)|^2\,\mathrm{d}x. \tag{2.9}$$

Remark 16. The fact that kinetic energy never dissipates in a fluid with no viscosity means that the system may sustain perpetual motion. For example, a vortex solving the Euler equations never stops spinning. Set $d=2$ and use polar coordinates (r,θ) and the associated basis (e_r, e_θ), defined by $e_r := (\cos(\theta), \sin(\theta))$ and $e_\theta = (-\sin(\theta), \cos(\theta))$. Then, for any even function $\phi\in C^\infty(\mathbb{R})$, the vector field

$$u = \phi(r)e_\theta$$

is a time independent solution of the Euler equations (2.4) associated to some pressure π. We let the reader check this: it is a matter of showing that the curl of $(u\cdot\nabla)u$ is zero, so that it must be equal to the gradient of some function. This computation becomes impossible as soon as there is energy dissipation through viscosity.

We may rephrase the energy conservation law (2.9) in terms of L^2 norms. Although this is a completely trivial rewriting, it is a conceptually crucial step. We have

$$\sup_{t\in\mathbb{R}_+}\|u(t)\|_{L^2} = \|u_0\|_{L^2}.$$

In other words, the solution defines a function of time $t\longmapsto L^2(\mathbb{R}^d)$ that is bounded. It lies in the space $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^d))$. This remarks motivates the following definition.

Definition 17. *Consider an interval $I\subset\mathbb{R}$ and $p,q\in[1,+\infty]$. We define $L^p_I(L^q) := L^p(I; L^q(\mathbb{R}^d))$ to be the set of measurable functions $f: I\times\mathbb{R}^d\longrightarrow\mathbb{R}$ such that*

$$\|f\|_{L^p_I(L^q)} := \left(\int_I \|f(t)\|^p_{L^q}\,\mathrm{d}t\right)^{1/p} < +\infty$$

wirh an obvious modification if $p=+\infty$. Note that, if the norm above is finite, the function $t\longmapsto\|f(t)\|_{L^q}$ is well-defined and finite for almost every $t\in I$ by the Fubini-Tonelli theorem.

For the sake of simplicity, we will use the following shorthand:

- For spaces of functions defined on the whole Euclidean space, we will omit the dependency on $\mathbb{R}^d$. For example, we write $L^p(\mathbb{R}^d) = L^p$.
- We will note $L^p(\mathbb{R}_+; L^q) = L^p(L^q)$, and it will always be assumed that we work with nonnegative times.
- When $I = [0, T[$, we note $L^p(I; L^q) = L^p_T(L^q)$.

Remark 18. Note that Definition 17 can be trivially extended to Sobolev type spaces, such as $L^p(H^N)$. We will also work with spaces of continuous functions defined on a time interval and with values in a Banach space X, such as $C^0(L^2)$.

The reader is reminded that the Sobolev space H^N is the space of all functions $f \in L^2$ such that $\nabla^k f \in L^2$ for all $0 \leq k \leq N$ (the derivative is taken in the sense of distributions). The space H^N is a Banach space when associated to the norm

$$\|f\|_{H^N} = \|f\|_{L^2} + \|\nabla^N f\|_{L^2}.$$

It can easily be proved that $H^M \subset H^N$ whenever $N \leq M$ (use the Fourier transform). Therefore, the H^N norm is equivalent to

$$\|f\|_{H^N} \approx \sum_{k=0}^{N} \|\nabla^k f\|_{L^2}.$$

We refer to the excellent book [5] for a cristal clear introduction to H^N spaces, starting with the simpler one dimensional case.

2.4 Existence and Uniqueness of Solutions

In this paragraph, we prove the main theorem of this chapter: local well-posedness of the Euler equations. We will first state the Theorem, and then turn to the different steps of the proof. As this is an *introduction* to mathematical hydrodynamics, the result we present is highly non-optimal: this is due to the fact that we try to use methods that are as elementary as possible, leaving more sophisticated tools for later.

Theorem 19 (Ebin-Marsden 1970 [13])**.** *Consider $d \geq 2$ and $N > d + 1$. For any initial datum $u_0 \in H^N$ such that $\mathrm{div}(u_0) = 0$, there is a time $T > 0$ such that the Euler equations (2.4) have a solution u in the space $L^\infty_T(H^N) \cap W^{1,\infty}_T(H^2)$, which is associated to a pressure $\pi \in L^\infty_T(L^2)$.*

On the other hand, any solution $v \in L^\infty_T(H^N) \cap W^{1,\infty}_T(H^2)$ associated to some pressure $\pi \in C^0_T(\mathcal{S}')$ with the same initial datum must be equal to u. In other words, the solution to the initial value problem is unique in that space.

Remark 20. If X is a Banach space, the space $W^{1,\infty}(X)$ is the space of $\mathbb{R}_+ \longrightarrow X$ functions that are Lipschitz: $f \in W^{1,\infty}(X)$ if and only if

$$\forall t_1, t_2 \in \mathbb{R}_+, \qquad \|f(t_2) - f(t_1)\|_X \leq K|t_2 - t_1|$$

for some constant K. It can be shown (the proof is not that hard) that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitz if and only if its (distributional) derivative is bounded $f' \in L^\infty$, hence the notation $W^{1,\infty}$ (as $W^{N,p}$ is the Sobolev space of functions f with $\nabla^k f \in L^p$ for all $0 \leq k \leq N$).

The proof of Theorem 19 is made of three steps. The first one is, essentially a formal computation: given a very regular initial datum, we try to find out how long the solutions is expected to keep H^N regularity. The second step is the actual construction of solutions, which involves approximating system (2.4) by a simple ODE. Finally, we have to prove that the solution we have constructed is the only one solving the Cauchy problem.

Before proving the theorem, let us point out that the original proof of Ebin and Marsden is based on flow map techniques: estimates that are centered on the particle trajectories in the fluid. The proof we present here is rather based on energy estimates (a.k.a. L^2-type estimates), and is very close to that of T. Kato [21], although with a few simplifications.

Also, we should say something about the improvements that could be made to Theorem 19. To start with, our assumptions require initial data in H^N with $N \geq d+2$. The proof of Kato in [21] shows that this condition can be relaxed to $N > 1 + d/2$. Next, it is possible to relax further the assumptions on the initial data by using different spaces containing H^N (for example the fractional Sobolev space $H^{1+d/2+\epsilon}$). The best result today (that we know of) is due to Pak and Park [25] who prove existence and uniqueness for u_0 belonging to the Besov space $B^1_{\infty,1}$, which is a space that is somewhat in between Lipschitz and C^1-Hölder functions: $C^{1,\epsilon} \subset B^1_{\infty,1} \subset W^{1,\infty}$. In two dimensions $d = 2$, this can even be further improved to include initial data u_0 that are log-Lipschitz. We refer to the classical article of Yudovich [26], or to the recent preprint [12] for ideas on how this could work.

2.4.1 *A Priori* Estimates

As the reader will probably have deduced from all the pages above, understanding a PDE problem passes through a lot of formal computations before one can actually dive into rigorous mathematical proof, especially when trying to prove well-posedness. This is due to the very nature of these results: the main point is finding the right function space X such that any initial datum $u_0 \in X$ will give a (possibly unique) solution in a space $u \in Y$. But how are we to guess what X and Y can be, especially as we do not yet know a solution exists?

The choice of the spaces X and Y is certainly not arbitrary, it depends entirely on the PDE and its properties. For example, the energy conservation laws (2.9) above suggest that if the initial datum is $u_0 \in L^2$, then the solution of the Euler system is in $u \in L^\infty(L^2)$, in turn suggesting as a possible choice $X = L^2$ and $Y = L^\infty(L^2)$.

In fact, the energy spaces $X = L^2$ and $Y = L^\infty(L^2)$ are far too weak for our purposes: because we will use compact embeddings (such as the Rellich-Kondrachov theorem), working in spaces of functions with higher regularity, H^N in this instance, is a must. The question is whether it is possible to pick Y to be such a space. In other words, if u_0 is regular enough, can we show that the solution will retain some regularity?

Therefore, the first step of our analysis is rather formal: we consider a regular solution u of the Euler equations, associated to a smooth integrable (and divergence-free) initial datum u_0, and wonder whether it is possible to show that $\|u(t)\|_{H^N}$ is finite, at least on some time interval. Because these computations are done before we have even showed the existence of a solution, they are called *a priori* estimates. We fix a multi-index $\alpha \in \mathbb{N}^d$ with length $|\alpha| = N$, and study the quantity $\|\partial^\alpha u\|_{L^2}$. We have[5]

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int \frac{1}{2} |\partial^\alpha u|^2 \, \mathrm{d}x &= \int \partial^\alpha u \cdot \partial_t \partial^\alpha u \, \mathrm{d}x \\ &= -\int \partial^\alpha \big(u_k \partial_k u\big) \cdot \partial^\alpha u \, \mathrm{d}x - \int \partial^\alpha \nabla \pi \cdot \partial^\alpha u \, \mathrm{d}x. \end{aligned}$$

First of all, note that the function $\partial^\alpha u$ is a divergence-free vector field $\mathrm{div}(\partial^\alpha u) = 0$ by the Schwarz theorem. Therefore, integrating by parts inside the second integral shows that

$$-\int \partial^\alpha \nabla \pi \cdot \partial^\alpha u \, \mathrm{d}x = \int \partial^\alpha \pi \, \mathrm{div}(\partial^\alpha u) \, \mathrm{d}x = 0.$$

[5] If $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ is a multi-index, the α-th derivative is defined by $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}$. We define $|\alpha| = \alpha_1 + \cdots + \alpha_d$ to be the length of the multi-index.

In order to study the variation of $\|\partial^\alpha u\|_{L^2}$, we therefore only have the first integral to contend with. Here, we expand the derivative by using an "improved binomial formula". If we set

$$\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} := \frac{\alpha!}{(\alpha-\gamma)!\,\gamma!}, \qquad \text{where} \quad \begin{array}{l} \alpha! := \alpha_1! \cdots \alpha_d! \\ \gamma! := \gamma_1! \cdots \gamma_d!, \end{array}$$

then we have[6]

$$\begin{aligned} \int \partial^\alpha(u_k\partial_k u)\cdot\partial^\alpha u\,\mathrm{d}x &= \sum_{|\gamma|\le|\alpha|} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \int \partial^\gamma u_k \partial^{\alpha-\gamma}\partial_k u \cdot \partial^\alpha u\,\mathrm{d}x \\ &:= \sum_{|\gamma|\le|\alpha|} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} I_\gamma. \end{aligned} \tag{2.10}$$

The sum above bears on all multi-indices $\gamma \in \mathbb{N}^d$ of length $|\gamma| \le |\alpha| = N$. Now, our goal will be to bound the integrals I_γ by using only the quantity $E(t) = \|u\|_{H^N}$. If we can manage this, we will have obtained a differential inequality of the form $E'(t) \le F\big(E(t)\big)$, on which Grönwall-type techniques will work. There are three cases.

First case: $\gamma = 0$. At a first glance, this is the most worrisome situation. We want to bound I_0 by using only $\|u\|_{H^N}$, but the integral involves a $(N+1)$-th derivative $\partial^\alpha\partial_k u$. However, we are saved by an integration by parts trick: the integral of a function of the form $(v\cdot\nabla)f\cdot f$ is always zero if $\mathrm{div}(v) = 0$, because it is a total derivative $v\cdot\nabla(\frac{1}{2}|f|^2)$. In other words, thanks to the divergence-free condition $\mathrm{div}(u) = 0$,

$$\begin{aligned} I_0 = \int u_k\partial_k\partial^\alpha u\cdot\partial^\alpha u\,\mathrm{d}x &= -\int \partial^\alpha u\cdot\partial_k\big(u_k\partial^\alpha u\big)\,\mathrm{d}x \\ &= -\int \partial^\alpha u\cdot u_k\partial_k\partial^\alpha u\,\mathrm{d}x = 0. \end{aligned}$$

The integral I_0 is zero, and we are left with integrals which involve less than N derivatives on every term.

Second case: $1 \le |\gamma| < N - \frac{d}{2}$. Here, the essential tool to bound I_γ is a Sobolev embedding. As a reminder, we state and prove the following lemma.

Lemma 21. *Consider an integer $k > \frac{d}{2}$. Then we have a continuous inclusion $H^k \subset L^\infty$. More precisely, if $f \in H^k$, then $f \in L^\infty$ and $\|f\|_{L^\infty} \le C(k,d)\|f\|_{H^k}$.*

Proof. The proof is simple and relies on the Fourier transform. By using the Plancherel identity, notice that

$$\|f\|_{H^k}^2 = \int \Big(|f|^2 + |\nabla^k f|^2\Big)\,\mathrm{d}x = C(d)\int \big(1+|\xi|^{2k}\big)|\widehat{f}(\xi)|^2\mathrm{d}\xi.$$

Therefore, the function $(1+|\xi|)^k|\widehat{f}(\xi)|$ is L^2. Secondly, the Fourier transform of any L^1 function is bounded, and any function whose Fourier transform is L^1 must be a bounded function. By writing this for f, we have

$$\|f\|_{L^\infty} \le C(d)\|\widehat{f}\|_{L^1} = C(d)\int \frac{1}{\big(1+|\xi|\big)^k}\big(1+|\xi|\big)^k|\widehat{f}(\xi)|\mathrm{d}\xi,$$

[6] While this multi-index formulation may seem intimidating, we will not use it that much: the main point is that it is possible to distribute derivatives inside a product by a Leibnitz-type rule, with given coefficients that depend only on the order of the derivatives. In other words, the exact form of (2.10) is not important, the main point is that such a formula *exists*.

so that the Cauchy-Schwarz inequality provides

$$\begin{aligned}\|f\|_{L^\infty} &\le C(d)\left(\int \frac{\mathrm{d}\xi}{\left(1+|\xi|\right)^{2k}}\right)^{1/2}\left(\int \left(1+|\xi|\right)^{2k}|\widehat{f}(\xi)|^2\mathrm{d}\xi\right)^{1/2}\\ &\le C(d)\|f\|_{H^k}.\end{aligned}$$

All the integrals above are finite because we have assumed $k > d/2$. This ends the proof of the lemma. □

Let us make use of this Sobolev embedding. By a simple Cauchy-Schwarz inequality, we may bound the integral I_γ by

$$\begin{aligned}|I_\gamma| &\le \|\partial^\alpha u\|_{L^2}\|\partial^\gamma u_k\|_{L^\infty}\|\partial^{\alpha-\gamma}\partial_k u\|_{L^2}\\ &\le \|u\|_{H^N}\|\nabla^{|\gamma|}u\|_{L^\infty}\|\nabla^{N-|\gamma|+1}u\|_{L^2}.\end{aligned}$$

Then Lemma 21 provides $\|\nabla^{|\gamma|}u\|_{L^\infty} \le C(d,N)\|\nabla^{|\gamma|}u\|_{H^k} \le \|u\|_{H^{k+|\gamma|}}$ for all $k > d/2$. And because we have assumed that $|\gamma| < N - \frac{d}{2}$, we may chose k so that $k+|\gamma| \le N$. Finally, we have the bound

$$|I_\gamma| \le C(d,N)\|u\|_{H^N}^3. \tag{2.11}$$

Third case: $N - \frac{d}{2} \le |\gamma| \le N$. This last case is very similar to the previous one. We start by using Hölder's inequality to obtain

$$\begin{aligned}|I_\gamma| &\le \|\partial^\alpha u\|_{L^2}\|\partial^\gamma u_k\|_{L^2}\|\partial^{\alpha-\gamma}\partial_k u\|_{L^\infty}\\ &\le \|u\|_{H^N}^2\|\nabla^{N-|\gamma|+1}u\|_{L^\infty}.\end{aligned}$$

Using again the Sobolev embedding of Lemma 21, we may bound the L^∞ norm above by

$$\|\nabla^{N-|\gamma|+1}u\|_{L^\infty} \le C(d,N)\|\nabla^{N-|\gamma|+1}u\|_{H^k} \le C(d,N)\|u\|_{H^{N+k+1-|\gamma|}}\ ,$$

and this inequality holds for any $k > d/2$. In order for us to bound this above by using the H^N norm only, we require that $k+1-|\gamma| \le 0$, or, by our current assumption on $|\gamma|$,

$$1+k \le N - \frac{d}{2}.$$

This inequality is fulfilled as soon as $N > d+1$, which is exactly the assumption of Theorem 19. With this hypothesis, we have $\|u\|_{H^{N+k+1-|\gamma|}} \le \|u\|_{H^N}$, so that I_γ also satisfies inequality (2.11). Overall, we have proven that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha u\|_{L^2}^2 \le C(d,N)\|u\|_{H^N}^3.$$

By adding this tho the energy conservation law (2.9) and simplifying a $\|u\|_{H^N}$ factor, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^N} \le C(d,N)\|u\|_{H^N}^2.$$

This is a differential inequality solved by $\|u\|_{H^N}$. As we have explained above, the goal is to use Grönwall-type methods in order to obtain "natural" bounds on the solution: assuming that the initial datum is in a given space $u_0 \in X$ (here $X = H^N$), we want to show that the solution naturally lies in a space $u \in Y$ (where Y is to be determined). In other words, we want to find a space Y such that $\|u\|_Y \leq \|u_0\|_X$. Let us integrate the inequality above in order to obtain an expression involving the initial datum u_0. We have

$$\|u(t)\|_{H^N} \leq \|u_0\|_{H^N} + C(d,N)\int_0^t \|u(s)\|_{H^N}^2 \mathrm{d}s.$$

Our goal is to find an upper bound which involves only the norm of the initial datum. The main obstacle is the time integral in the righthand side. However, this integral will be very small for small times, so that it should be possible to bound explicitly $\|u\|_{H^N}$ by the initial datum on a short time interval. With this in mind, we define

$$T = \sup\left\{t \geq 0, \quad C(d,N)\int_0^t \|u(s)\|_{H^N}\mathrm{d}s \leq \|u_0\|_{H^N}\right\},$$

so that on the interval $[0,T]$, we have the desired estimate $\|u(t)\|_{H^N} \leq 2\|u_0\|_{H^N}$. We may rephrase this inequality as $\|u\|_{L^\infty_T(H^N)} \leq C(d,N)\|u_0\|_{H^N}$. Of course, this is not entirely satisfying: while we have bounded the solution only by the initial datum, we have done so on a time interval... which is defined in terms of the solution! (and not the initial datum) To correct this, we try to estimate the time T by a function of $\|u_0\|_{H^N}$. Notice that, for all $t \in [0,T]$, we must have

$$C(d,N)\int_0^t \|u(s)\|_{H^N}^2 \mathrm{d}s \leq 4C(d,N)t\|u_0\|_{H^N}^2.$$

Therefore, any time such that $2C(d,N)t\|u_0\|_{H^N} \leq \|u_0\|_{H^N}$ must be lower than T, hence the lower bound

$$T \geq \frac{C(d,N)}{\|u_0\|_{H^N}}. \tag{2.12}$$

Let us summarize: we have shown that for $N > 2 + d/2$, it is possible to bound the solution by using the initial datum only:

$$\|u\|_{L^\infty_T(H^N)} \leq 2\|u_0\|_{H^N},$$

where the time T can also be bounded from below by a function of $\|u_0\|_{H^N}$. As we will see, the space $L^\infty_T(H^N)$ has much better properties that the energy space $L^\infty(L^2)$, in particular with regards to compact embeddings.

2.4.2 Construction of a Solution

In this paragraph, we show the existence of a solution u to the Cauchy problem with initial datum u_0. The idea is to see u as a limit of approximate solutions: functions u_n which solve a sequence of approximate equations, that is systems which look very much alike to the Euler equations, but that essentially behave as solutions of an ODE.

STEP 1: approximate system. We start by defining an approximation operator: for any $n \geq 1$, we define a Fourier multiplier A_n by the formula

$$\forall f \in L^2, \qquad \widehat{A_n f}(\xi) = \mathbb{1}_{|\xi|\leq n}\widehat{f}(\xi).$$

Note that the operator A_n is the L^2-orthogonal projections on functions that are spectrally supported in the ball $\{|\xi| \leq n\}$. In particular, for any $f \in L^2$, the function $A_n f$ is smooth (analytic in fact), while we have the property

$$A_n f \underset{n\to+\infty}{\longrightarrow} f \qquad \text{in } L^2. \tag{2.13}$$

With the help of the approximation operator A_n, we may write the following approximate Cauchy problem:

$$\begin{cases} \partial_t u_n + A_n \mathbb{P}\big((u_n \cdot \nabla) u_n\big) = 0 \\ u_n(0) = A_n u_0. \end{cases} \tag{2.14}$$

STEP 2: approximate solution. We start by showing that the approximate system (2.14) has has a solution. The idea here is to consider (2.14) as an ODE problem and to apply the Cauchy-Lipschitz theorem. We therefore need to show that the map

$$f \longmapsto -A_n \mathbb{P}\big((f \cdot \nabla) f\big)$$

is locally Lipschitz in a well-chosen Banach space X_n. We choose $X_n = A_n(L^2)$ to be the space of all L^2 functions whose Fourier transform is supported n the ball $B(0, n)$, in other words,

$$X_n = \Big\{ f \in L^2, \quad \operatorname{supp}(\widehat{f}) \subset B(0, n) \Big\},$$

which we equip with the $\| \, . \, \|_{L^2}$ norm. Note that X_n functions are C^∞ (analytic in fact). We have, for any $f, g \in X_n$, by the Plancherel identity,

$$\big\| A_n \mathbb{P}\big((f \cdot \nabla) g\big) \big\|_{L^2}^2 = C(d) \int_{|\xi| \leq n} |\xi|^2 \big|\sigma(\xi)\big|^2 \left| \widehat{f \cdot \nabla g}(\xi) \right|^2 \mathrm{d}\xi,$$

where $\sigma(\xi)$ is the symbol of the operator $\mathbb{P}$ and is a homogeneous function of degree zero. Because this integral is defined on frequencies $|\xi| \leq n$, we may write

$$\begin{aligned} \big\| A_n \mathbb{P}\big((f \cdot \nabla) g\big) \big\|_{L^2}^2 &\leq C(d) n^2 \|(f \cdot \nabla) g\|_{L^2} \\ &\leq C(d) n^2 \|f\|_{L^2} \|\nabla g\|_{L^\infty} \\ &\leq C(d) n^2 \|f\|_{L^2} \int |\xi| \big|\widehat{g}(\xi)\big| \mathrm{d}\xi. \end{aligned}$$

Hölder's inequality applied to the integral above (which is supported on $|\xi| \leq n$) yields

$$\big\| A_n \mathbb{P}\big((f \cdot \nabla) g\big) \big\|_{L^2}^2 \leq C(d) n^{3+d/2} \|f\|_{L^2} \|g\|_{L^2}.$$

As a consequence, we see that

$$\big\| A_n \mathbb{P}\big((f \cdot \nabla) f\big) - A_n \mathbb{P}\big((g \cdot \nabla) g\big) \big\|_{L^2} \leq \|f\|_{L^2} \|f - g\|_{L^2} + \|g\|_{L^2} \|f - g\|_{L^2},$$

so that the map is indeed locally Lipschitz in the space X_n. By use of the Cauchy-Lipschitz theorem, we see that problem 2.14 has a unique maximal solution $u_n \in C^1([-T_n^*, T_n[; X_n)$ for some $T_n^*, T_n > 0$.

STEP 3: approximate solution lifespan. We have constructed a family of approximate solutions (u_n) whose (forward) lifespan T_n may depend on the approximation index $n \geq 1$. Let us show that this is not the case and that $T_n = +\infty$. By taking the product of the ODE in (2.14) by u_n and integrating with respect to space, we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int |u_n|^2 \, \mathrm{d}x + \int u_n \cdot A_n \mathbb{P}\big((u_n \cdot \nabla) u_n\big) \, \mathrm{d}x = 0.$$

Now, we notice that, for any given time $t \in [0, T_n[$, the function $u_n(t)$ is both divergence free (because $u_n(0)$ is and $\operatorname{div}(\partial_t u_n) = 0$) and has its Fourier transform supported in the ball $|\xi| \leq n$.

Therefore, $A_n\mathbb{P}u_n = u_n$. By using the fact that the symbol $\sigma(\xi)$ is a symmetric matrix (recall from Remark 15 that $\mathbb{P}$ is an orthogonal projection), we have that

$$\begin{aligned}\int u_n \cdot A_n\mathbb{P}\big((u_n\cdot\nabla)u_n\big)\,\mathrm{d}x &= \int \widehat{u_n}(\xi)\cdot \mathbb{1}_{|\xi|\leq n}\sigma(\xi).\widehat{u_n\cdot\nabla u_n}(\xi)\mathrm{d}\xi\\ &= \int \mathbb{1}_{|\xi|\leq n}\sigma(\xi).\widehat{u_n}(\xi)\cdot\widehat{u_n\cdot\nabla u_n}(\xi)\mathrm{d}\xi\\ &= \int u_n\cdot(u_n\cdot\nabla)u_n\,\mathrm{d}x,\end{aligned}$$

and this integral is zero, by the computations (2.8) in which we established the conservation of energy.[7] In summary, we have shown that the ODE solutions remains on the surface of a sphere of the space X_n,

$$\|u_n(t)\|_{X_n} = \|A_nu_0\|_{X_n}, \tag{2.15}$$

and so it cannot blow-up in finite time, thanks to the following lemma.

Lemma 22. *Consider a Banach space X and a locally Lipschitz map $F: X\longrightarrow X$. For $y_0\in X$, let $y :]T_-, T_+[\longrightarrow X$ be the unique maximal solution of the ODE $y' = F(y)$ with initial datum $y(0) = y_0$. Then either $T_+ = +\infty$ or*

$$\overline{\lim_{t\to T_+}}\|y(t)\|_X = +\infty.$$

Proof. Assume on the contrary that $T_+ < +\infty$ and that the solution y remains bounded as $t\to T_+$. Then, the family $\big(y(t)\big)_{t<T_+}$ satisfies a Cauchy criterion:

$$\forall\, T_- < s < t < T_+, \qquad \big\|y(t)-y(s)\|_X \leq |t-s|\sup_{]s,T_+[}\big\|F\big(y(\tau)\big)\big\|_X.$$

The supremum in the equation above is finite because y is bounded on a neighborhood of T_+ and F is continuous. We deduce that $y(t)$ has a limit $y_+\in X$ as $t\to T_+$. But if z is the maximal solution of the ODE with datum $z(T_+) = y_+$, then z allows to extend continuously y beyond T_+. Because the extension is continuous and satisfies the integral formulation of the ODE, it must also be C^1, and so is a solution of the ODE that contradicts the maximality of y. □

From the above, we deduce that the lifespan of the approximate solutions is infinite $T_n = +\infty$. In particular, it does not depend on n.

STEP 4: construction of a weak limit. All this shows that we have a family of approximate solutions $u_n\in C^1(\mathbb{R}_+;X_n)$. Now, we want to make the approximation parameter tend to infinity $n\to+\infty$ and hope to recover, in the limit, a solution of the Euler equations. On the one hand, the computations immediately above show that the sequence (u_n) lies within a fixed ball of the space $L^\infty(L^2)$, as

$$\|u_n\|_{L^\infty(L^2)} = \|A_nu_0\|_{L^2}\leq\|u_0\|_{L^2},$$

this upper bound being independent on n. This uniform bound already allows us to find a limit to the sequence (u_n). Because the space $L^\infty(L^2)$ has a separable predual $(L^1(L^2))' = L^\infty(L^2)$, the Banach-Alaoglu theorem (Theorem 3.16 and Corollary 3.30 in [5], see also Theorem 75 in the appendix below) provides an extraction $\phi:\mathbb{N}\to\mathbb{N}$ and a weak limit $u\in L^\infty(L^2)$ such that[8]

$$u_{\phi(n)}\overset{*}{\rightharpoonup} u \qquad \text{in } L^\infty(L^2).$$

[7]If this feels a bit uncomfortable, the reader may check that integrals of this type are always zero if u_n is a Schwartz function $u_n\in\mathcal{S}$. Then, remark that functions whose Fourier transform is C^∞ and are compactly supported in the ball $B(0,n)$ are Schwartz and form a dense subset of X_n.

[8]It is quite possible that the reader is unfamiliar or newly acquainted with the notions of weak or weak-($*$) convergence. An explanatory appendix (Appendix A) dedicated to the presentation of these ideas through examples is located at the end of these notes.

However, it is not obvious at all that u is a solution of the Euler problem. For that, we would have to be able to let $n \to +\infty$ in the approximate problem (2.14) and recover the Euler equations. The issue is the non-linear term: because the convergence above is *weak*, it is not a given that $A_n\mathbb{P}(u_n \cdot \nabla u_n)$ converge to $\mathbb{P}(u \cdot \nabla u)$. We need to show that the sequence (u_n), or an extraction thereof, converges in a *stronger* topology.

STEP 5: high order estimates. Recall $N \geq d+2$ from the statement of Theorem 19 we are trying to prove, and fix a multi-index $|\alpha| = N$. Then, by taking the α-th space derivative of the Approximate equation (2.14), we obtain, after integrating by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial^\alpha u_n|^2 \,\mathrm{d}x + \int \partial^\alpha u_n \cdot \partial^\alpha A_n\mathbb{P}\big((u_n \cdot \nabla)u_n\big)\,\mathrm{d}x = 0.$$

Once again, the function $\partial^\alpha u_n$ is both divergence-free and has its Fourier transform supported in a ball $|\xi| \leq n$, so that $A_n\mathbb{P}\partial^\alpha u_n = u_n$. This means that the approximate solution satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial^\alpha u_n|^2 \,\mathrm{d}x + \int \partial^\alpha u_n \cdot \partial^\alpha (u_n \cdot \nabla)u_n\,\mathrm{d}x = 0,$$

which is exactly the form of the trilinear integral (2.10) we had bound in order to find *a priori* estimates! By repeating *mutatis mutandi* the computations of Subsection 2.4.1, we find that

$$\|u_n(t)\|_{H^N} \leq \|A_n u_0\|_{H^N} + C(d,N)\int_0^t \|u_n(s)\|_{H^N}^2 \mathrm{d}s \tag{2.16}$$

so that the approximate solutions u_n fulfill the bound

$$\|u_n(t)\|_{H^N} \leq 2\|A_n u_0\|_{H^N} \leq 2\|u_0\|_{H^N}$$

as long as the time t is smaller than

$$t \leq T := \frac{C(d,N)}{\|u_0\|_{H^N}} \leq \frac{C(d,N)}{\|A_n u_0\|_{H^N}}.$$

We have shown that the sequence (u_n) entirely lies within a ball (of radius $2\|u_0\|_{H^N}$) of the space $L^\infty_T(H^N)$.

Remark 23. The paragraph above is absolutely typical in the study of non-linear PDEs. One first starts by looking for formal *a priori* estimates, knowing that they will usually transfer to the approximate solutions without too much difficulty. Although the construction of a solution is quite technical, involving tools of functional analysis, the real core of the work usually lies within the *a priori* estimates.

STEP 6: strong compactness. In order to prove strong compactness properties on the sequence (u_n), we will use the Ascoli theorem[9]. Because many different versions of this result exist, we state one that will be useful in our context and give a short proof.

Theorem 24. *Consider a compact metric space (K,d) and a compact interval $I \subset \mathbb{R}$. Let (f_n) be a sequence of functions in $C^0(I;K)$ that is uniformly equicontinuous: for all $\epsilon > 0$, we may fix a $\delta > 0$ such that*

$$\forall t, s \in I, \forall n \geq 1, \qquad |t-s| \leq \delta \Rightarrow d\big(f_n(t), f_n(s)\big) \leq \epsilon. \tag{2.17}$$

Then, there exists an extraction $\phi : \mathbb{N} \to \mathbb{N}$ such that the sequence $(f_{\phi(n)})$ converges in the space $C^0(I;K)$.

[9] This result is also known as the *Arzelà-Ascoli theorem*. It seems Ascoli has given a sufficient condition for compactness while Arzelà found a necessary one. The theorem has been further improved by many authors, and we will actually be using a variant of it in our proof.

Proof. Let us start by constructing a candidate accumulation point of the sequence (f_n). Consider a countable set of points $\mathcal{T} := \{t_k,\ k \geq 1\}$ which is dense in I. Because K is compact, we can fix, for any $k \geq 1$, an extraction $\phi_k : \mathbb{N} \to \mathbb{N}$ such that the sequence $\big(f_{\phi_k(n)}(t_k)\big)_n$ is convergent:

$$f_{\phi_k(n)}(t_k) \underset{n\to+\infty}{\longrightarrow} g_k \qquad \text{in } K.$$

By diagonal extraction, we construct a composite extraction $\phi : \mathbb{N} \to \mathbb{N}$ such that, for every $k \geq 1$,

$$f_{\phi(n)}(t_k) \underset{n\to+\infty}{\longrightarrow} g_k \qquad \text{in } K. \tag{2.18}$$

Now, let us define a function $g : \mathcal{T} \longrightarrow K$ by setting $g(t_k) = g_k$. We will show that g is continuous (for the topology of $\mathcal{T}$ induced by the absolute value). Consider $\epsilon > 0$. By assumption, we may fix a $\delta > 0$ such that the equicontinuity implication (2.17) holds. Consequently, for any $k, \ell \geq 1$ such that $|t_k - t_\ell| \leq \delta$ and any $n \geq 1$, we have $d(f_n(t_k), f_n(t_\ell)) \leq \epsilon$. By taking the limit $n \to +\infty$ in this inequality, we see that

$$d\big(f_n(t_k), f_n(t_\ell)\big) \underset{n\to+\infty}{\longrightarrow} d\big(g(t_k), g(t_\ell)\big) \leq \epsilon, \tag{2.19}$$

and this proves that g is continuous. The density of $\mathcal{T}$ in I means that g has a unique continuous extension to I, which we still note g.

In order to finish the proof, we have to show that the sequence $(f_{\phi(n)})$ does indeed converge to g in the space $C^0(I, K)$. Consider once again a $\epsilon > 0$ and a $\delta > 0$ such that (2.17) holds, and (2.19) shows that the same equicontinuity inequality also is true for g. In addition, applying the Borel-Lebesgue property to the density of $\mathcal{T}$ and the compactness of I allows us to fix a rank k_0 such, for any $t \in I$, there is a t_k lying in the finite set $\mathcal{T}_0 := \{t_k,\ 1 \leq k \leq k_0\}$ with $|t - t_k| \leq \delta$. For any $t \in I$, we may fix such a t_k and write

$$\begin{aligned} d\big(f_{\phi(n)}(t), g(t)\big) &\leq d\big(f_{\phi(n)}(t), f_{\phi(n)}(t_k)\big) + d\big(f_{\phi(n)}(t_k), g(t_k)\big) + d\big(g(t_k), g(t)\big) \\ &\leq 2\epsilon + d\big(f_{\phi(n)}(t_k), g(t_k)\big). \end{aligned}$$

Thanks to (2.18), we know that the distance in the upper bound converges to zero as $n \to +\infty$, and the finiteness of $\mathcal{T}_0$ means that this convergence is uniform with respect to k. There exists a rank $M \geq 1$ that is independent of k such that $d\big(f_{\phi(n)}(t_k), g(t_k)\big) \leq \epsilon$ for all $t_k \in \mathcal{T}_0$. This ends the proof of uniform convergence. ☐

Our aim is now to apply this theorem to the sequence of approximate solutions (u_n), which is bounded in the space $L^\infty_T(H^N)$. For this, we must show that it possesses a equicontinuity property with respect to time and has its values in a compact set. In order to obtain equicontinuity in the time variable, we will seek an estimate on the time derivative $\partial_t u_n$. We have:

$$\|\partial_t u_n\|_{H^2} = \big\|A_n \mathbb{P}((u_n \cdot \nabla)u_n)\big\|_{H^2}.$$

Now, two remarks are in order. Firstly, for any vector field $f \in H^k$ with $k \geq 0$, we have, by the Plancherel identity,

$$\begin{aligned} \|A_n \mathbb{P} f\|^2_{H^k} &\approx \sum_{|\alpha|\leq k} \|\partial^\alpha A_n \mathbb{P} f\|^2_{L^2} \\ &\approx \sum_{|\alpha|\leq k} \int_{|\xi|\leq n} |\xi|^{2|\alpha|} \big|m(\xi).\widehat{f}(\xi)\big|^2 \mathrm{d}\xi \\ &\leq C(d) \sum_{|\alpha|\leq k} \|\partial^\alpha f\|^2_{L^2} \leq C(d)\|f\|^2_{H^k}, \end{aligned}$$

so that the Leray projection operator is bounded on H^k. In particular, we have

$$\|\partial_t u_n\|_{H^2} \leq C(d)\big\|(u_n \cdot \nabla)u_n\big\|_{H^2}.$$

Secondly, we note that the functions $(u_n \cdot \nabla)u_n$ are uniformly bounded in $L^\infty_T(H^2)$. This comes from the following estimate: if $j, k \in \{1, ..., d\}$ are two indices, then the Sobolev embedding of Lemma 21 shows that

$$\begin{aligned}
\big\|\partial_j\partial_k(u_n \cdot \nabla u_n)\big\|_{L^2} &\leq \|\partial_j\partial_k u_n\|_{L^2}\|\nabla u_n\|_{L^\infty} + \|\partial_j u_n\|_{L^\infty}\|\partial_k \nabla u_n\|_{L^2} + \|\partial_k u_n\|_{L^\infty}\|\partial_j \nabla u_n\|_{L^2} \\
&\qquad + \|u_n\|_{L^\infty}\|\partial_j\partial_k \nabla u_n\|_{L^2} \\
&\leq C\|u_n\|_{H^3}\big(\|u_n\|_{L^\infty} + \|\nabla u_n\|_{L^\infty}\big) \\
&\leq C(d)\|u_n\|^2_{H^N},
\end{aligned}$$

since the regularity exponent is larger than $N > 2+d/2 \geq 3$. Putting these two remarks together, we obtain a bound on the time derivative which is uniform with respect to the approximation parameter:

$$\|\partial_t u_n\|_{L^\infty_T(H^2)} \leq C(d)\|u_n\|^2_{L^\infty_T(H^N)} \leq C(d)\|u_0\|^2_{H^N}.$$

In particular, the sequence (u_n) is bounded in the space $W^{1,\infty}_T(H^2)$. Now, by invoking the Rellich-Kondrachov (Theorem 3.16 in [5], see also Theorem 63 in Appendix A below), we know that the space H^2 is compactly embedded in the space H^1_{loc}, which is the space of locally H^1 functions[10] associated to the metric

$$d(f, g) := \sum_{k=0}^{\infty} \frac{1}{2^k} \min\left\{1, \big\|(f-g)\chi_k\big\|_{H^1}\right\}, \tag{2.20}$$

where $\chi_k(x) = \chi(x/k)$ and χ is a smooth, nonnegative and compactly supported function that has value $\chi = 1$ on the unit ball $B(0,1) \subset \mathbb{R}^d$. This makes it possible to invoke our variant of Ascoli's theorem, as a Lipschitz condition is stronger than equicontinuity and as a bounded part of H^2 is a compact subset of H^1_{loc}. Theorem 24 therefore provides an extraction $\phi : \mathbb{N} \to \mathbb{N}$ such that we have convergence

$$u_{\phi(n)} \longrightarrow u \qquad \text{in } C^0_T(H^1_{\mathrm{loc}}). \tag{2.21}$$

In addition, applying the Banach-Steinhaus theorem (a.k.a. the uniform boundedness principle, Theorem 2.2 in [5]) also shows that the limit u lies in all the spaces in which we had uniform bounds: $u \in L^\infty_T(H^N) \cap W^{1,\infty}_T(H^2)$.

STEP 7: taking the limit. Now, we wish to take the limit $n \to +\infty$ in the approximate system. We already have a family of approximate solutions $(u_{\phi(n)})$ that converges strongly (2.21) to a function $u \in L^\infty(H^N)$. These functions solve the approximate system

$$\partial_t u_{\phi(n)} + A_{\phi(n)}\mathbb{P}\big((u_{\phi(n)} \cdot \nabla)u_{\phi(n)}\big) = 0$$

On the one hand, convergence of the time derivative in the sense of distributions is straightforward. Let $\psi \in \mathcal{D}(]0, T[\times\mathbb{R}^d)$ be a smooth and compactly supported function. We may fix a compact set $K \subset \mathbb{R}^d$ such that $\mathrm{supp}(\psi) \subset]0, T[\times K$. Therefore,

$$\begin{aligned}
\big\langle \partial_t u_{\phi(n)}, \psi\big\rangle &= -\int_0^T \int u_{\phi(n)} \cdot \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t \\
&\longrightarrow -\int_0^T \int u \cdot \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t = \langle \partial_t u, \psi\rangle.
\end{aligned}$$

[10] The local aspect of the convergence is optimal: imagine a nonzero function $f \in H^2(\mathbb{R})$ and a sequence $f_n(x) = f(x-n)$. Then the sequence (f_n) converges locally to zero, but does not do so in H^1 norm.

In other words, we have shown that

$$\partial_t u_{\phi(n)} \longrightarrow \partial_t u \qquad \text{in } \mathcal{D}'(]0,T[;\mathbb{R}^d).$$

On the other hand, we take care of the product $(u_n \cdot \nabla)u_n$ in the following way: for any compact subset $K \subset \mathbb{R}^d$, we have

$$\big\|(u_{\phi(n)}\cdot\nabla)u_{\phi(n)} - (u\cdot\nabla)u\big\|_{L^2(K)} \leq \|u_{\phi(n)}-u\|_{L^2(K)}\|\nabla u_{\phi(n)}\|_{L^\infty(K)} + \|u\|_{L^\infty(K)}\|\nabla u_{\phi(n)}-\nabla u\|_{L^2(K)},$$

and the uniform bound (2.16) with the Sobolev embedding of Lemma 21 then provide

$$\begin{aligned}\big\|(u_{\phi(n)}\cdot\nabla)u_{\phi(n)} - (u\cdot\nabla)u\big\|_{L^2(K)} &\leq C(d,N)\|u_{\phi(n)}-u\|_{L^2(K)}\|u_0\|_{H^N} \\ &\qquad + C(d,N)\|u\|_{L^\infty(K)}\|\nabla u_{\phi(n)}-\nabla u\|_{L^2(K)} \\ &\longrightarrow 0 \qquad \text{as } n \to +\infty.\end{aligned}$$

This shows that the bilinear convective term $(u_{\phi(n)} \cdot \nabla)u_{\phi(n)}$ converges to $(u\cdot\nabla)u$ in the L^2_{loc} topology. Unfortunately, the local aspect of this convergence creates a few problems, due to the fact that the operators $A_{\phi(n)}$ and $\mathbb{P}$ are precisely *non-local*. For example, the Leray projection operator $\mathbb{P}$ is not bounded (not to mention ill-defined) on the space L^2_{loc}. We circumvent this problem by resorting to weak convergence. Since the functions $(u_n\cdot\nabla)u_n$ are uniformly bounded in the Banach space $L^\infty_T(L^2)$, whose predual is the separable space $L^1_T(L^2)$, we may invoke the Banach-Alaoglu theorem (Theorem 75 in Appendix A) to show the sequence has an accumulation point $g \in L^\infty_T(L^2)$ for the weak-$(*)$ convergence. This means that, up to an (omitted) extraction, the convergence

$$\begin{aligned}\big\langle (u_{\phi(n)}\cdot\nabla)u_{\phi(n)}, \psi\big\rangle_{L^\infty_T(L^2)\times L^1_T(L^2)} &= \int_0^T\int (u_{\phi(n)}\cdot\nabla)u_{\phi(n)}\cdot\psi\,\mathrm{d}x\,\mathrm{d}t \\ &\longrightarrow \int_0^T\int g\cdot\psi\,\mathrm{d}x\,\mathrm{d}t = \big\langle g,\psi\big\rangle_{L^\infty_T(L^2)\times L^1_T(L^2)}\end{aligned}$$

holds for any function $\psi \in L^1_T(L^2)$. In particular, for compactly supported ψ, the L^2_{loc} convergence of the sequence shows that $g = (u\cdot\nabla)u$. We deduce the weak-$(*)$ convergence

$$(u_{\phi(n)}\cdot\nabla)u_{\phi(n)} \overset{*}{\rightharpoonup} (u\cdot\nabla)u \qquad \text{in } L^\infty_T(L^2). \tag{2.22}$$

Loosely stated, the advantage of this weak-$(*)$ convergence over the L^2_{loc} one is that it is, in some sense, "less local", that is not limited to compact sets. Let us exploit this for our problem. We fix again a function $\psi \in L^1_T(L^2)$ and write

$$\begin{aligned}\big\langle A_{\phi(n)}\mathbb{P}\big((u_{\phi(n)}\cdot\nabla)u_{\phi(n)}\big),\psi\big\rangle &= \big\langle\big((u_{\phi(n)}\cdot\nabla)u_{\phi(n)}\big), A_{\phi(n)}\mathbb{P}\psi\big\rangle \\ &= \big\langle\big((u_{\phi(n)}\cdot\nabla)u_{\phi(n)}\big) - (u\cdot\nabla)u, \mathbb{P}\psi\big\rangle \\ &\qquad + \big\langle\big((u_{\phi(n)}\cdot\nabla)u_{\phi(n)}\big), \big(A_{\phi(n)}-\mathrm{Id}\big)\mathbb{P}\psi\big\rangle + \big\langle\big((u\cdot\nabla)u,\mathbb{P}\psi\big\rangle\end{aligned}$$

Now, the function $\mathbb{P}\psi$ belongs to $L^1_T(L^2)$, as the operator $\mathbb{P}$ is bounded $L^2 \longrightarrow L^2$. The weak-$(*)$ convergence (2.22) then shows that the first of the three brackets in the righthand side above converges to zero. In addition, equation (2.13) shows that $A_{\phi(n)}\mathbb{P}\psi$ converges to $\mathbb{P}\psi$ strongly in $L^1_T(L^2)$. This follows from the fact that

$$\big\|\big(A_{\phi(n)}-\mathrm{Id}\big)\mathbb{P}\psi\big\|_{L^1_T(L^2)} = \int_0^T \big\|\big(A_{\phi(n)}-\mathrm{Id}\big)\mathbb{P}\psi(t)\big\|_{L^2}\,\mathrm{d}t,$$

and that the quantities $\big\|\big(A_{\phi(n)}-\mathrm{Id}\big)\mathbb{P}\psi(t)\big\|_{L^2}$ converge to zero for almost every time, by (2.13), so dominated convergence provides the convergence to zero of the whole $L^1_T(L^2)$ norm. We deduce convergence of the second bracket:

$$\begin{aligned}\big|\big\langle\big((u_{\phi(n)}\cdot\nabla)u_{\phi(n)}\big),\big(A_{\phi(n)}-\mathrm{Id}\big)\mathbb{P}\psi\big\rangle\big| &\leq \big\|(u_{\phi(n)}\cdot\nabla)u_{\phi(n)}\big\|_{L^\infty_T(L^2)}\big\|\big(A_{\phi(n)}-\mathrm{Id}\big)\mathbb{P}\psi\big\|_{L^1_T(L^2)}\\ &\leq \|u_0\|^2_{H^N}\big\|\big(A_{\phi(n)}-\mathrm{Id}\big)\mathbb{P}\psi\big\|_{L^1_T(L^2)}\\ &\longrightarrow 0.\end{aligned}$$

Overall, we have proven that

$$A_{\phi(n)}\mathbb{P}\big((u_{\phi(n)}\cdot\nabla)u_{\phi(n)}\big)\overset{*}{\rightharpoonup}\mathbb{P}\big((u\cdot\nabla)u\big)\qquad\text{in } L^\infty_T(L^2).$$

We deduce that the limit u must fulfill the Euler equations with Leray projection in the sense of distributions:

$$\partial_t u+\mathbb{P}\big((u\cdot\nabla)u\big)=0\qquad\text{in } \mathcal{D}'(]0,T[\times\mathbb{R}^d). \tag{2.23}$$

Note that this implies that the solution is divergence-free $\mathrm{div}(u)=0$, provided that the initial datum also is $\mathrm{div}(u_0)=0$, as we have $\partial_t\,\mathrm{div}(u)=0$.

STEP 8: the pressure. In Step 7, we have proved that the function $u\in L^\infty_T(H^N)$ is a solution to the projected problem, this is equation (2.23). However, in order to prove that u is a solution of the *Euler equations*, we must show the existence of a pressure function π with

$$\partial_t u+(u\cdot\nabla)u+\nabla\pi=0.$$

Recall the expression of the Leray projection operator: $\mathbb{P}=\mathrm{Id}+\nabla(-\Delta)^{-1}\,\mathrm{div}$. In other words, we must find a function π such that

$$\nabla\pi=\nabla(-\Delta)^{-1}\,\mathrm{div}\,\big((u\cdot\nabla)u\big). \tag{2.24}$$

Here, we recall a simple computation from the introduction, which relies on the divergence-free property of the solution. We have

$$\begin{aligned}\partial_j\partial_k(u_ju_k)&=\partial_j\big(u_k\partial_ku_j+u_j\partial_ku_k\big)\\ &=\mathrm{div}\,\big((u\cdot\nabla)u\big).\end{aligned}$$

Therefore, we want to define the function π by the identity

$$\pi:=(-\Delta)^{-1}\partial_j\partial_k(u_ju_k). \tag{2.25}$$

Of course, it is not the only choice possible: adding any function $f(t)$ of time only would also yield a pressure fulfilling equation (2.24). Let us show that our definition (2.25) is correct. For that, we must understand it in the sense of Fourier multipliers: we rigorously define π by the identity

$$\widehat{\pi}(\xi):=-\frac{\xi_j\xi_k}{|\xi|^2}\widehat{u_ju_k}(\xi).$$

Because of the Sobolev embedding of Lemma 21, we know that $H^N\subset L^\infty$, so that the Fourier Plancherel identity gives

$$\|\pi\|_{L^2}\leq C(d)\|u\otimes u\|_{L^2}\leq C(d)\|u\|_{L^\infty}\|u\|_{L^2}\leq C(d)\|u\|^2_{H^N},$$

and we see that π is well-defined as an element of the space $L^\infty_T(L^2)$.

STEP 9: initial datum. The last step to prove that u is a solution of the initial value problem is to check that it is associated to the correct initial datum. Recall from (2.21) that the approximate solutions can be shown to converge uniformly with respect to time:

$$u_{\phi(n)}\longrightarrow u\qquad\text{in } C^0_T(H^1_{\mathrm{loc}}).$$

In particular, convergence also happens at the initial time $t=0$, where we have

$$u_{\phi(n)}(0)=A_{\phi(n)}u_0\longrightarrow u_0\qquad\text{in } L^2,$$

and so, by uniqueness of the limit, we deduce that $u(0)=u_0$, and that u is indeed a solution of the initial value problem for the Euler equations. This ends the proof of existence.

2.4.3 Uniqueness of Solutions

In this paragraph, we show that the solution we have constructed is the only possible solution of the initial value problem on the time interval $[0, T[$. This relies on a stability estimate: we consider to initial data $u_{0,1}, u_{0,2} \in H^N$ associated to two solutions $u_1, u_2 \in L^\infty_T(H^N) \cap W^{1,\infty}_T(H^2)$ associated to pressures $\pi_1, \pi_2 \in C^0_T(\mathcal{S}')$, on some common time interval $T > 0$, and we show that the solutions u_1 and u_2 cannot be to far apart if the initial data are close enough. In other words, we want to show that the initial datum to solution map is continuous, which implies uniqueness of the solution.

We define the difference function $\delta u := u_2 - u_1$, and define δu_0 in the same way. The difference function solves a PDE involving u_1 and u_2, namely

$$\partial_t(\delta u) + (u_2 \cdot \nabla)u_2 - (u_1 \cdot \nabla)u_1 + \nabla(\pi_2 - \pi_1) = 0,$$

and $\mathrm{div}(\delta u) = 0$. By letting the difference function δu appear in the equation above, we obtain

$$\begin{cases} \partial_t(\delta u) + (u_2 \cdot \nabla)\delta u + (\delta u \cdot \nabla)u_1 + \nabla(\delta\pi) = 0 \\ \mathrm{div}(\delta u) = 0, \end{cases}$$

where we have of course set $\delta\pi = \pi_2 - \pi_1$. Now the temptation is very great to multiply the first equation by δu and integrate, just as we did when we were estimating the evolution of the kinetic energy, or the H^N norms. If we did that, we would get, after integrating with respect to time

$$\int_0^t \int \delta u \cdot \partial_t(\delta u)\, \mathrm{d}x\mathrm{d}s + \int_0^t \int (u_2 \cdot \nabla)\delta u \cdot \delta u\, \mathrm{d}x\mathrm{d}s + \int_0^t \int (\delta u \cdot \nabla)u_1 \cdot \delta u\, \mathrm{d}x\mathrm{d}s + \int_0^t \int \nabla(\delta\pi) \cdot \delta u\, \mathrm{d}x\mathrm{d}s = 0. \tag{2.26}$$

However, before using such an equation, we have to be very careful as to whether all the integrals are properly defined. While this was no problem when we were doing *a priori* estimates, which we then applied to the very regular approximate solutions, the functions u_1 and u_2 cannot be assumed to possess all the required regularity. Here, formal computations will not do. For example, it is not at all clear that the pressure functions π_1 and π_2 have enough regularity for the last integral to exist: while we have constructed a solution with a $L^\infty_T(L^2)$ pressure[11], nothing indicates that π_1 and π_2 share that regularity.

Let us start by noticing that, thanks to the assumptions on the solutions $u_1, u_2 \in L^\infty_T(H^N) \cap W^{1,\infty}_T(H^2)$, the first three integrals in (2.26) are well-defined. On top of that, concerning the first integral, but we also see that the equation

$$\int_0^t \int \delta u \cdot \partial_t(\delta u)\, \mathrm{d}x\mathrm{d}s = \int_0^t \int \frac{1}{2}\partial_t |\delta u|^2\, \mathrm{d}x\mathrm{d}s = \frac{1}{2}\int |\delta u(t)|^2\, \mathrm{d}x - \frac{1}{2}\int |\delta u_0|^2\, \mathrm{d}x$$

must hold for all times $t \in [0, T[$. Indeed, it does for regular functions, and all elements of this equation are well-defined and continuous functions of u in the $W^{1,1}_T(H^2)$ topology, which agree on a dense subset of that space.[12]

On the other hand, the pressure integral is not obviously well-defined, as we are yet unsure as to the regularity of $\delta\pi$. This is because, in Theorem 19, we want to focus on the regularity of the

[11] The pressure is in fact much better than that: it can be proven that it is in fact in $L^\infty_T(H^{N+1})$, but that would take us too far for the time being.

[12] Because $\delta u \in W^{1,\infty}_T(H^2)$, it would be quite tempting to invoke a density argument in that space. However, as my students have wisely attracted my attention to this point, the spaces of regular functions in the time variable, such as $C^1_T(H^2)$, are *not* dense in $W^{1,\infty}_T(H^2)$. Remember that the space $W^{1,\infty[0,T]}$ is not separable: it is isometric to $L^\infty[0, T]$.

velocity field and emphasize the fact the the regularity of the pressure is secondary. But doing so comes at a cost: we now must make sure that the pressure cannot be too singular. This is the purpose of the following proposition.

Proposition 25. *Consider a solution $u \in L^\infty_T(H^N) \cap W^{1,\infty}_T(H^2)$ of the Euler equations associated to the pressure $\pi \in C^0_T(\mathcal{S}')$. Then there is a function of time $f \in C^0([0,T[;\mathbb{R}^d)$ such that we have $\pi - f(t) \in L^\infty(L^2)$ and*

$$\pi = f(t) + (-\Delta)^{-1}\partial_j\partial_k(u_j u_k).$$

Proof. In order to study the pressure, we take the divergence of the first equation in the Euler system: as we have done before, we obtain an elliptic equation (2.5) solved by the pressure:

$$-\Delta\pi = \partial_j(u_k\partial_k u_j) = \partial_k\partial_k(u_j u_k).$$

By assumption, we know that the pressure π is at all times a tempered distribution. In terms of Fourier transform, this means that the pressure satisfies the equality

$$|\xi|^2\widehat{\pi}(\xi) = -\xi_j\xi_k\widehat{u_j u_k}(\xi).$$

so that we must have

$$\widehat{\pi}(\xi) = -\frac{\xi_j\xi_k}{|\xi|^2}\widehat{u_j u_k}(\xi) := \widehat{\pi_0}(\xi) \qquad \text{for } \xi \neq 0.$$

In other words, the equation above holds in the sense of distributions *away from* $\xi = 0$, that is in $\mathcal{D}'(]0,T[\times\mathbb{R}^d \setminus \{0\})$. This implies that the Fourier transforms $\widehat{\pi}(t)$ and $\widehat{\pi_0}(t)$ must agree up to the addition of a distribution supported at $\xi = 0$. Such a distribution is necessarily a linear combination of the Dirac delta δ_0 and its derivatives (see for example Proposition 2.4.1 in [19]), and so is the Fourier transform of a polynomial function $Q(t) \in \mathbb{R}[X]$. In summary, we have shown that, for all times $t \in [0,T[$,

$$\pi(t) = Q(t) + (-\Delta)^{-1}\partial_j\partial_k\big(u_j(t)u_k(t)\big), \tag{2.27}$$

and plugging this in the equation gives

$$\partial_t u + \mathbb{P}\operatorname{div}(u\otimes u) + \nabla Q = 0.$$

Now, remember that $u \in W^{1,\infty}_T(H^2)$. Consequently, we must also have, for all times $t \in [0,T[$,

$$\nabla Q(t) = -\partial_t u - \mathbb{P}\operatorname{div}(u\otimes u) \in L^\infty_T(L^2) \in L^2 \cap \mathbb{R}[X].$$

But since the only polynomial function in L^2 is zero, this implies that $Q(t)$ is of degree at most zero: it is a constant polynomial. We may conclude by noting that (2.27) gives the desired conclusion: $\pi = Q(t) + \pi_0$, the continuity of Q as a function of time following from that of π_0 and π. □

Remark 26. The proof of Proposition 25 can probably be simplified in the setting of $L^\infty_T(H^N) \cap W^{1,\infty}_T(H^2)$ solutions. However, we felt it was important to include it, as it is generalizable to other settings where the regularity of the pressure field is hard to ascertain. We refer to [10] for a discussion of this.

A consequence of Proposition 25 is that the pressure integral is well-defined (as a distribution bracket) and equal to zero: the equation

$$\int_0^t\int \nabla(\delta\pi)\cdot\delta u \,\mathrm{d}x\mathrm{d}s = -\int_0^t\int \delta\pi\operatorname{div}(\delta u)\,\mathrm{d}x\mathrm{d}s = 0$$

holds for all regular functions δu, and all parts of the equation are continuous maps of δu with respect to the $L^\infty_T(H^1)$ topology. By density, it is true because $\delta u = u_2 - u_1 \in L^\infty_T(H^2)$.

We may justify in the same way the integration by parts that takes care of the second integral in (2.26). The identity

$$\int_0^t \int (u_2 \cdot \nabla)\delta u \cdot \delta u \,\mathrm{d}x\mathrm{d}s = -\int_0^t \int (u_2 \cdot \nabla)\delta u \cdot \delta u \,\mathrm{d}x\mathrm{d}s = 0$$

holds for $\delta u \in L^\infty_T(H^2)$ because both members of the first equation are continuous maps of δu in the $L^2_T(H^1)$ topology.

Putting everything together, we obtain the following:

$$\int |\delta u|^2 \,\mathrm{d}x = \int |\delta u_0|^2 \,\mathrm{d}x + \int_0^t \int (\delta u \cdot \nabla) u_1 \cdot \delta u \,\mathrm{d}x\mathrm{d}s,$$

and Hölder's inequality gives

$$\|\delta u\|_{L^2}^2 \leq \|\delta u_0\|_{L^2} + \int_0^t \|\delta u\|_{L^2} \|\nabla u_1\|_{L^\infty},$$

so that an application of Grönwall's lemma and the Sobolev embedding of Lemma 21 yields the stability estimate we were seeking:

$$\|\delta u\|_{L^\infty_T(L^2)} \leq \|\delta u_0\|_{L^2} \exp\Big(T \|u_1\|_{L^\infty_T(H^N)}\Big). \tag{2.28}$$

In particular, if $u_{0,1} = u_{0,2}$, then we must have $u_1 = u_2$ almost everywhere on $[0, T[\times \mathbb{R}^d$. This ends the proof of the local well-posedness of the Euler equations, Theorem 19.

Remark 27. The L^2 stability inequality (2.28) is essentially optimal: the takeaway is that if the solutions are not Lipschitz (or possibly log-Lipschitz, or something of the kind), there is no hope to prove uniqueness of solutions. This should be compared with ODEs of the type $y' = f(y)$, where uniqueness of solutions fail when the function f is not Lipschitz (or generally does not fulfill the Osgood condition). The comparison is not gratuitous: particle trajectories ϕ_t in an ideal fluid satisfy an ODE $\partial_t \phi_t = u(t, \phi_t)$ involving the solution. If the velocity field u is not locally Lipschitz, then the particle flow will not be uniquely defined.

2.5 Blow-Up Criterion and Possible Breakdown of Regularity

In the previous section, we have shown that any initial datum $u_0 \in H^N$ can be associated to a unique local solution $u \in L^\infty_{\mathrm{loc}}([0, T_N[; H^N)$, whose lifespan is at least

$$T_N \geq \frac{C(d, N)}{\|u_0\|_{H^N}}. \tag{2.29}$$

This type of lower bound for the lifespan is consistent with the quadratic nature of the Euler equations: $\partial_t u$ is equal to the function $\mathbb{P}\,\mathrm{div}(u \otimes u)$, which is quadratic with respect to u. As an illustration, look at the ODE $y' = y^2$, which can be solved explicitly for any initial datum $y_0 \in \mathbb{R}$, and which gives solutions that blow-up at time $T^* = 1/y_0$.

While this is reassuring, several questions remain. Firstly, while this lower bound for the lifespan is consistent with the quadratic nature of the equation, we have not proven that H^N solutions do indeed blow-up. In other words, we do not know if the lifespan is finite $T_N < +\infty$, so that

$$\|u(t)\|_{H^N} \longrightarrow +\infty \qquad \text{as } t \to T_N.$$

In fact, this is a very hard question. While it is solved in two dimensions $d = 2$ (solutions do not blow-up), the answer is still missing in three dimensions $d = 3$.

Another question that naturally rises from inequality (2.29) is that of the nature of the blow-up. Assume that $u_0 \in H^N$ for some high value of N. So far, we have to keep in mind that the lifespan of the solution $u \in L^\infty_{\mathrm{loc}}([0, T_N[; H^N)$ may depend on N. In other words, the solution may loose H^N regularity at time T_N while retaining some lower regularity, say H^M, up to some time T_M. The solutions would then progressively loose regularity. Sciences

Of course, this is only one of two possible outcomes. Another possibility is that at time T_N the solution abruptly looses all its regularity, and becomes discontinuous. In that case, the lifespan of the H^N solution would be the same as the one of the H^M solution and $T_N = T_M$. Given that this is precisely the way solutions blow-up in the case of Burgers equation, this scenario might seem the most likely.

The goal of this section is to find out what precisely happens at the blow-up time. For this, we will introduce the notion of fractional regularity and use it to prove an important interpolation inequality.

Remark 28. The reader may be wondering whether a strong solution that blows-up transforms into a (non-unique) global weak solution, which would presumably lie in the energy space $C^0(L^2)$. This interesting question turns out to be challenging in the extreme.

2.5.1 Fractional Sobolev Spaces and Interpolation Inequalities

As we have hinted in the introduction to this section, fractional regularity is a concept that may be of some importance in the study of non-linear PDEs. For example, if it turns out a solution of a non-linear PDE looses regularity as time goes by, we would expect this process to be continuous. The solution might be initially H^2, and then at time T be H^1. But what about an intermediate time $0 < t < T$? Would the solution be $H^{3/2}$ at some point? How should we make sense of this space?[13]

There are many other very good reasons to introduce fractional regularity spaces, most of which are not covered in these notes: trace theory, achieving optimal results, finding scale-critical spaces, interpolation theory, *etc.* For the time being, we focus on interpolation inequalities. Let us start by defining fractional Sobolev spaces.

Definition 29. *Consider $s \in \mathbb{R}$. We define H^s to be the space of all $f \in \mathcal{S}'$ such that $\widehat{f}$ is locally integrable and*

$$\|f\|_{H^s} := \left(\int \left| (1 + |\xi|^2)^{s/2} \widehat{f}(\xi) \right|^2 \mathrm{d}\xi \right)^{1/2} < +\infty.$$

The space H^s is Banach. Moreover, if $s \in \mathbb{N}$, the Fourier-Plancherel identity shows that this definition of H^s coincides with the usual one, expressed in terms of the L^2 norms of f and $\nabla^s f$.

One of the very great advantages of the space H^s is that it is particularly well-suited for handling Fourier multipliers. For example, consider the operator $(\mathrm{Id} - \Delta)^{s/2}$ which is of order $s \in \mathbb{R}$ and is defined by its Fourier transform. Then we have

$$\|f\|_{H^s} = \left\| (\mathrm{Id} - \Delta)^{s/2} f \right\|_{L^2}$$

for any $f \in H^s$. Likewise, the Leray projection operator is a bounded operator $\mathbb{P} : H^s \longrightarrow H^s$ for all $s \in \mathbb{R}$.

We now turn to the main reason why we introduce fractional spaces: interpolation inequalities.

[13] While we will see that this does *not* happen in the case of the Euler equations, this possibility is by no means completely theoretical, and explicit examples of such phenomena exist.

Proposition 30. *Consider two exponents $s_1 < s_2$ and $\theta \in [0,1]$. We set $s = \theta s_1 + (1-\theta)s_2$. Then, for any function $f \in H^{s_1} \cap H^{s_2}$, we have the inequality*

$$\|f\|_{H^s} \leq \|f\|_{H^{s_1}}^{\theta} \|f\|_{H^{s_2}}^{1-\theta}.$$

Remark 31. Such an inequality is called an *interpolation inequality*. The reader is already familiar with one such result, Hölder's inequality: consider $p < q$ and $\theta \in [0,1]$ and define r by

$$\frac{1}{r} := \frac{\theta}{p} + \frac{1-\theta}{q},$$

Then we have $\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta} \|f\|_{L^q}^{1-\theta}$ for all $f \in M^p \cap L^q$. In particular, $L^r \subset L^p \cap L^q$.

Proof. The proof is very straightforward. For convenience, we define the japanese bracket

$$\forall \xi \in \mathbb{R}^d, \qquad \langle \xi \rangle := \big(1 + |\xi|^2\big)^{1/2}.$$

Then,

$$\begin{aligned}
\|f\|_{H^s}^2 &= \int \langle \xi \rangle^{2s} \left| \widehat{f}(\xi) \right|^2 \mathrm{d}\xi \\
&= \int \langle \xi \rangle^{2\theta s_1} \left| \widehat{f}(\xi) \right|^{2\theta} \langle \xi \rangle^{2(1-\theta)s_2} \left| \widehat{f}(\xi) \right|^{2(1-\theta)} \mathrm{d}\xi,
\end{aligned}$$

and applying Hölder's inequality with exponents $p = \frac{1}{\theta}$ and $p' = \frac{1}{1-\theta}$, we have

$$\begin{aligned}
\|f\|_{H^s}^2 &\leq \left(\int \langle \xi \rangle^{2s_1} \left| \widehat{f}(\xi) \right|^2 \mathrm{d}\xi \right)^{\theta} \left(\int \langle \xi \rangle^{2s_2} \left| \widehat{f}(\xi) \right|^2 \mathrm{d}\xi \right)^{1-\theta} \\
&= \left(\|f\|_{H^{s_1}}^{\theta} \|f\|_{H^{s_2}}^{1-\theta} \right)^2.
\end{aligned}$$

□

Sobolev embeddings may also be written with fractional Sobolev spaces. For example, the following lemma is proved exactly in the same way as Lemma 21.

Lemma 32. *Consider $s > d/2$. Then we have a continuous inclusion $H^s \subset L^\infty$. More precisely, if $f \in H^s$, then $f \in L^\infty$ and $\|f\|_{L^\infty} \leq C(s,d)\|f\|_{H^s}$.*

Remark 33. The Sobolev embedding $H^s \subset L^\infty$ is *not* true in the critical case where $s = d/2$. For example, the reader can check that the function $\log|x|$ lies within $H^1(\mathbb{R}^2)$. Instead, the critical Sobolev space $H^{d/2}$ is a part of the space BMO of functions of Bounded Mean Oscillations, but that is a whole different topic.

2.5.2 A Closer Look at the *A Priori* Estimates

In this paragraph, we use the interpolation inequalities from just above to revisit the *a priori* estimates. Recall that, for a multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| = N$, we had

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int |\partial^\alpha u|^2 \,\mathrm{d}x &= -\sum_{\gamma \leq \alpha} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \int \partial^\alpha u \cdot \partial^\gamma u_k \partial_k \partial^{\alpha-\gamma} u \,\mathrm{d}x \\
&:= -\sum_{\gamma \leq \alpha} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} I_\gamma.
\end{aligned}$$

We had shown through integration by parts that the integral for γ was $I_0 = 0$. For the others, we had distinguished between two cases, according to whether it was possible or not to bound

$\partial^\gamma u_k$ in L^∞ by using Sobolev embeddings. Consider a $\sigma > d/2$ whose precise value we will fix later, and which is not necessarily an integer, so that we have $H^\sigma \subset L^\infty$ by Lemma 32. Assume in addition that $N > d+1$, as in Theorem 19.

First case: if $k := |\gamma| \le N - \sigma$, then we may write, thanks to Lemma 32,

$$\begin{aligned}|I_\gamma| &\le \|\nabla^N u\|_{L^2} \|\nabla^k u\|_{L^\infty} \|\nabla^{N-k+1} u\|_{L^2} \\ &\le C(d,N,\sigma) \|u\|_{H^N} \|u\|_{H^{\sigma+k}} \|u\|_{H^{N+1-k}}.\end{aligned}$$

By assumption on $1 \le k \le N - \sigma$, all the regularity exponents appearing in the inequality above are smaller than N. When we were working on the *a priori* estimates the first time, we had simply bounded all the norms above by the H^N norm, which is grossly non-optimal. This time, we will instead use the interpolation inequality of Proposition 30 to have better inequalities. Consider $\theta, \theta' \in [0,1]$ such that

$$\begin{aligned}\sigma + k &= (\sigma+1)\theta + N(1-\theta) \\ N + 1 - k &= (\sigma+1)\theta' + N(1-\theta'),\end{aligned}$$

or in other words

$$\theta = \frac{N-\sigma-k}{N-\sigma-1} \qquad \text{and} \qquad \theta' = \frac{k-1}{N-\sigma-1}.$$

In particular, note that $\theta + \theta' = 1$. Then Proposition 30 gives

$$\begin{aligned}|I_\gamma| &\le C(d,N,\sigma)\|u\|_{H^N} \|u\|_{H^{\sigma+1}}^{\theta} \|u\|_{H^N}^{1-\theta} \|u\|_{H^{\sigma+1}}^{\theta'} \|u\|_{H^N}^{1-\theta'} \\ &\le C(d,N,\sigma) \|u\|_{H^{\sigma+1}} \|u\|_{H^N}^2.\end{aligned}$$

Second case: we may perform exactly the same operation when $k > N - \sigma$. The Sobolev embedding $H^\sigma \subset L^\infty$ once again yields

$$\begin{aligned}|I_\gamma| &\le \|\nabla^N u\|_{L^2} \|\nabla^k u\|_{L^2} \|\nabla^{N-k+1} u\|_{L^\infty} \\ &\le C(d,N,\sigma) \|u\|_{H^N} \|u\|_{H^k} \|u\|_{H^{\sigma+N-k+1}}.\end{aligned}$$

By remembering that $N > d+1$, we see that the exponent $\sigma + N - k + 1 \le 2\sigma + 1$ may be chosen smaller than N provided that σ is chosen sufficiently close to $d/2$. We resort again to the interpolation inequalities of Proposition 30, but this time with $\theta, \theta' \in [0,1]$ chosen to that

$$\begin{aligned}k &= (\sigma+1)\theta + N(1-\theta) \\ \sigma + N + 1 - k &= (\sigma+1)\theta' + N(1-\theta'),\end{aligned}$$

or in other words,

$$\theta = \frac{N-k}{N-\sigma-1} \qquad \text{and} \qquad \theta' = \frac{k-\sigma-1}{N-\sigma-1},$$

so that, again, we have $\theta + \theta' = 1$. This implies that we have

$$|I_\gamma| \le C(d,N) \|u\|_{H^{\sigma+1}} \|u\|_{H^N}^2,$$

which is exactly the inequality we had obtained in the first case.

By summing with the energy conservation law (2.9) and over all multi-indices $|\alpha| = N$, we have shown that (for regular solutions), we must have the following differential inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{H^N}^2\right) \le C(d,N) \|u\|_{H^{\sigma+1}} \|u\|_{H^N}^2,$$

so that an application of Grönwall's lemma gives

$$\|u(t)\|_{H^N} \le \|u_0\|_{H^N} \exp\left(C(d,N) \int_0^t \|u(s)\|_{H^{\sigma+1}} \mathrm{d}s \right). \tag{2.30}$$

2.5.3 Discussion of the Blow-Up

In the previous paragraph, we have (formally) derived an inequality (2.30) which allows us to bound the $L_t^\infty(H^N)$ norm of the solution by using the $L_t^1(H^{\sigma+1})$ one. Assuming that the formal inequality (2.30) holds for any solution (we have not proven that it does), we may further our understanding of the blow-up. We had wondered what would happen to a maximal solution $u \in L^\infty_{\mathrm{loc}}([0, T_N[; H^N)$ at the blow-up time T_N. Does it experience loss of regularity and degenerate to a H^M solution with $M < N$, or is all the regularity lost at once, as was the case with Burgers equation? From inequality (2.30), we see that, in fact, as long as the solution keeps a $H^{\sigma+1}$ regularity, then it must stay as regular as the initial datum. A different way to say this is that a blow-up of the Euler equations happens (if it does happen at all) at all regularity levels at once.

As a consequence (this will be fully justifies in a moment), the lifespan of a solution $u \in L^\infty([0, T[; H^N)$ may be bounded from below by

$$T_N \geq \frac{C(d, N_0)}{\|u_0\|_{H^{N_0}}}, \tag{2.31}$$

regardless of how high $N \gg N_0$ is, provided that we have $N_0 > d + 1$. It would seem that, ultimately, the regularity of the initial datum plays only a minor role in the solution lifespan. For comparison, recall that the lifespan of a solution of the Burgers equation was $T^* = -1/\min \partial_x u_0$, and did not depend on how smooth u_0 is.

Next, we note that inequality (2.31) actually gives the rate at which the norm of the solution blows-up near its lifespan. If T^* is the finite lifespan of the solution u, given a time $0 \leq T^* - t < T^*$, we may consider the Euler problem with initial datum $u_{1,0} := u(T^* - t)$, which has a solution u_1 on a maximal interval $[0, T_1[$ with

$$T_1 \geq \frac{C(d)}{\|u_{1,0}\|_{H^{N_0}}} = \frac{C(d)}{\|u(t - T^*)\|_{H^{N_0}}}.$$

This new solution u_1 must coincide with u on their common lifespan, by uniqueness of H^{N_0} solutions. Therefore, its lifespan must be precisely $T_1 = t$. We deduce a blow-up rate inequality,

$$\|u(t)\|_{H^{N_0}} \geq \frac{C(d)}{T^* - t}. \tag{2.32}$$

Note that this inequality is consistent with (2.30), which states that the H^{N_0} norm cannot blow up unless the integral inside the exponential is infinite. This will surely happen at the blow-up time if inequality (2.32) holds.

Finally, we mention that (2.30) should be understood as a way to describe the blow-up of the maximal solution. If the solution $u \in L_T^\infty(H^N)$ has a finite lifespan $T^* < +\infty$, then the integral in the exponential must be infinite:

$$\int_0^{T^*} \|u(t)\|_{H^{\sigma+1}} \, \mathrm{d}t = +\infty.$$

If it were not, then it would be possible to (uniquely) extend the solution beyond T^*. In other words, the infinity of the integral above is a *blow-up criterion*. The actual argument is more complicated than its ODE analogue we saw in (22), but the main ideas are there. Overall, we have the following result, which we do not prove (time is lacking). The interested reader will find the general arguments for such results in Section 7.1.6 of [2] for example.

Theorem 34. *Consider $N > d + 1$ and any $\sigma > d/2$, an initial datum $u_0 \in H^N$ and $u \in L^\infty_{\mathrm{loc}}([0, T^*[; H^N) \cap W^{1,\infty}([0, T^*[; H^2)$ the unique maximal solution of the Euler problem associated to u_0. Then the lifespan T^* is finite if and only if*

$$\int_0^{T^*} \|u(t)\|_{H^{\sigma+1}} \, \mathrm{d}t = +\infty.$$

It is noteworthy that Theorem 34 is not optimal. The $H^{\sigma+1}$ condition therein can be replaced by the more general criterion

$$\int_0^{T^*} \|\mathrm{curl}\,(u)\|_{L^\infty}\,\mathrm{d}t = +\infty,$$

which is due to Beale-Kato and Majda (1984), and sometimes referred to as the BKM criterion for blow-up. We refer to [3] for the detail of their argument.

Chapter 3

Surface Quasi-Geostrophic Equations

In this chapter, we study a the existence and uniqueness of solutions for a different fluid problem, which will require the introduction of some additional mathematical methods. We will be focusing on the Surface Quasi-Geostrophic equation (SQG for short), which takes the following form:

$$\begin{cases} \partial_t\theta + u \cdot \nabla\theta = 0 \\ u = \nabla^\perp(-\Delta)^{-\beta/2}\theta. \end{cases} \tag{3.1}$$

In the above, the operator $\nabla^\perp$ is defined on scalar fields by the relation $\nabla^\perp f = (-\partial_2 f, \partial_1 f)$. The unknown θ is a scalar field, and will be considered on times $t \in \mathbb{R}$ and the space domain $x \in \mathbb{R}^2$. These equations are mainly used in geophysical sciences to describe the dynamics of Earth's atmosphere, and hence create weather forecasts. While we do not have the time to present a derivation of these equations (and instead refer to [18] pp. 18–29 for more explanations), we should still say that the unknown θ represents a potential temperature, that is the temperature the atmospheric gas would naturally have at a reference pressure level. This potential temperature is of course closely related to the actual temperature of the atmosphere, but its expression also involves the pressure and some thermodynamic compressibility coefficient. The vector field $u(t,x) \in \mathbb{R}^2$ is the velocity of the atmospheric fluid.

The operator $(-\Delta)^{-\beta/2}$ appearing in the equation is a *fractional Laplace* operator, or *fractional Laplacian*. It is defined in terms of Fourier transforms by the relation

$$\forall f \in \mathcal{S}, \qquad \widehat{(-\Delta)^{-\beta/2} f}(\xi) = |\xi|^{-\beta}\widehat{f}(\xi).$$

The presence of such an operator in a physical equation is due to the fact that equation (3.1) is a two dimensional model for what is really a three dimensional model. Such fractional operators frequently occur in trace problems, when one is trying to restrict the solution of a PDE defined on a half 3D space $\{x_3 > 0\}$ to a 2D problem set on the plane $\{x_3 = 0\}$. A precise example of this type of behavior is given by the Dirichlet to Neumann map.[1] The exponent $\beta \in]0,2[$ is linked to precise properties of the atmosphere.[2]

Finally, we should point out that SQG equations are part of a large family of PDEs, active scalar equations, that have attracted a lot of attention in the past years. The results of this chapter are (relatively) recent, and have been proved in [7].

[1] The Dirichlet to Neumann map on $\mathbb{R}^2 \times \{0\}$ is precisely the fractional Laplace operator $(-\Delta)^{1/2}$.

[2] The author of [18], pp. 27–28, convincingly argues that the full range of exponents $\beta \neq 2$ is relevant to geophysics, as these are linked Brunt-Väisälä frequencies that are altitude dependent (*e.g.* in an atmosphere where the potential temperature does not exactly have an exponential dependency on the altitude).

3.1 Transport Equations and SQG

We start by introducing the main difficulties of the SQG equation by a few comments on transport equations, which are the PDE analogue of ODEs. Consider a smooth vector field $u(t,x) \in \mathbb{R}^d$ which we will assume to have no divergence $\operatorname{div}(u) = 0$. For example, u can be thought as the velocity field of a fluid with constant density. The transport equation associated to u is the PDE

$$\partial_t f + u \cdot \nabla f = 0$$

equipped with the initial datum $f(0) = f_0$. Just as in the case of Burgers equation, a fruitful approach is to use the notion of characteristic curve: a curve $y(t) \in \mathbb{R}^d$ such that the quantity $f(t, y(t))$ is independent of time. Let us find those curves. Consider a smooth function $y(t)$ and set $g(t) = f(t, y(t))$. Then, we have

$$g'(t) = \partial_t f\big(t, y(t)\big) + y'(t) \cdot \nabla f\big(t, y(t)\big),$$

so that $y(t)$ is a characteristic curve as soon as is solves the ODE problem $y = u(t, y)$. With that in mind, we choose y to be the solution of the Cauchy problem

$$\begin{cases} y'(t) = u\big(t, y(t)\big) \\ y(0) = x \end{cases}$$

In other words, $y(t) = \phi_t(x)$ is the flow of the ODE $y' = u(t, y)$. We deduce that

$$f(t, \phi_t(x)) = g(t) = g(0) = f_0(x).$$

This provides an expression of the solution in terms of the inverse flow map of the ODE $y' = u(t, y)$, namely

$$f(t, x) = f_0\big(\phi_t^{-1}(x)\big).$$

While this expression is only explicit when we can compute the inverse flow map ϕ_t, it nevertheless shows that the transport equation is well-posed: it has exactly one solution stemming from a given initial datum. In addition, having an expression of some sorts of the solution also provides some information. In particular, the solution cannot be more regular than the flow ϕ_t or the initial datum is. For example, if ϕ_t is C^k but not C^{k+1}, then the solution at a given time $f(t)$ is C^k but cannot be C^{k+1}.

Now, we turn to the SQG equation and try to understand what this means about the regularity of solutions. Note that the velocity field u as defined in (3.1) has zero divergence, as $\operatorname{div}(\nabla^\perp) = 0$. Continuity properties of the operator $\nabla^\perp(-\Delta)$ are described in the following proposition.

Proposition 35. *Consider $\beta \leq 1$ and $s \in \mathbb{R}$. Then the Fourier multiplier $\nabla^\perp(-\Delta)^{-\beta/2}$ defines a continuous map in the $H^s \longrightarrow H^{s+\beta-1}$ topology.*

Proof. Consider $\theta \in L^1 \cap H^s$. By the Plancherel identity, we must prove that the Fourier function

$$\widehat{g}(\xi) = \langle \xi \rangle^{s+\beta-1} \frac{i\xi^\perp}{|\xi|^\beta} \widehat{\theta}(\xi)$$

is L^2. We simply note that the symbol $m(\xi) := i\xi^\perp / |\xi|^\beta$ is bounded around the origin $\xi = 0$ and is homogeneous of degree $1 - \beta$. Therefore,

$$\|\theta\|_{H^{s+\beta-1}}^2 = \int \langle \xi \rangle^{2(s+\beta-1)} |\xi|^{2(1-\beta)} \big|\widehat{\theta}(\xi)\big|^2 \, \mathrm{d}\xi \leq C(d) \int \langle \xi \rangle^{2s} \big|\widehat{\theta}(\xi)\big|^2 \, \mathrm{d}\xi = C \|\theta\|_{H^s}^2.$$

□

In order to illustrate the difficulty of our problem, we will assume that $\beta \in]0,1[$, so that the operator $\nabla^\perp(-\Delta)^{-\beta/2}$ is of positive order $1-\beta > 0$. Consider a solution associated to the initial datum $\theta_0 \in H^k$, and let ϕ_t be the flow map associated to the velocity field $u = \nabla^\perp(-\Delta)^{-\beta/2}\theta$. Then, because θ is a solution of a transport equation, it must satisfy

$$\theta(t,x) = \theta_0\big(\phi_t^{-1}(x)\big).$$

Our problems start here: because we have assumed the initial datum to have regularity H^k, this formula shows that we cannot expect the solution $\theta(t)$ to have more regularity than H^k at any given time $t \in \mathbb{R}$. But if $\theta(t) \in H^k$, then the velocity field has at best regularity

$$u(t) = \nabla^\perp(-\Delta)^{-\beta/2}\theta(t) \in H^{k+\beta-1} \not\subset H^k.$$

And if u is only $H^{k+\beta-1}$, then the flow cannot more than $H^{k+\beta-1}$, and in turn, θ is at best $H^{k+\beta-1}$. By repeating the argument *ad infinitum*, we see that the velocity field cannot be expected to possess any finite regularity H^s for any $s \in \mathbb{R}$, so that the solution might instantly collapse. The situation seems hopeless!

The observant reader will have noticed that we already have invoked such an argument to explain why Burgers equation ought to be ill-posed. But what had saved us was that Burgers equation possessed a helpful structure, linked to characteristic curves, which allowed it to be solved despite the initial pessimistic evaluation. In the case of SQG, we are trying to make the point that the transport equation structure (*a.k.a.* characteristic curves) cannot help study solutions. A new structure, more advanced than the simple use of characteristics, is needed to study the SQG equations.

We conclude this paragraph by a general lemma showing that the operator $A = \nabla^\perp(-\Delta)^{-\beta/2}$ is well-defined in the functional framework we will use.

Lemma 36. *Consider $0 < \beta < 2$ and $\epsilon > 0$. Then the operator $A : H^{2-\beta+\epsilon} \longrightarrow L^\infty$ is well-defined and bounded.*

Proof. Consider $f \in H^{2-\beta+\epsilon}$. Since $\|Af\|_{L^\infty}$ is bounded by the L^1 norm of the Fourier transform $\widehat{Af}$, we have

$$\|Af\|_{L^\infty} \leq C \int |\xi|^{1-2\beta}|\widehat{f}(\xi)|\mathrm{d}\xi = C \int \frac{|\xi|^{1-2\beta}}{\langle\xi\rangle^{2-\beta+\epsilon}}\langle\xi\rangle^{2-\beta+\epsilon}|\widehat{f}(\xi)|\mathrm{d}\xi.$$

Applying the Cauchy-Schwarz inequality to the above provides

$$\|Af\|_{L^\infty} \leq C \left(\int \frac{|\xi|^{2-2\beta}}{\langle\xi\rangle^{4-2\beta+2\epsilon}}\mathrm{d}\xi\right)^{1/2} \|f\|_{H^{2-\beta+\epsilon}}.$$

On the one hand, the function $|\xi|^{2-2\beta}$ is integrable on a neighborhood of $\xi = 0$ because $\beta < 2$. On the other hand, the function $|\xi|^{2-2\beta}\langle\xi\rangle^{-4+2\beta-\epsilon} = O\big(|\xi|^{-2-2\epsilon}\big)$ is integrable on $|\xi| \geq 1$. We deduce that the integral in the parenthesis is finite, and this proves the lemma. □

3.2 The Commutator Structure

The saving grace of the SQG equation is a hidden commutator structure that was first uncovered by F. Marchand in 2008 [24]. Consider a smooth solution of the SQG equation. By writing explicitly the value of the velocity field, we obtain the non-linear PDE

$$\partial_t\theta + \nabla^\perp(-\Delta)^{-\beta/2}\theta \cdot \nabla\theta = 0. \tag{3.2}$$

Now, we focus on finding an L^2 estimate. Of course, this can by using the transport structure of the PDE: multiplying and integrating by parts as in the energy estimates for the Euler equations yields $\|\theta(t)\|_{L^2} = \|\theta_0\|_{L2}$. But we will proceed in another way. For this, it is convenient to define the operators

$$(A_1, A_2) := \Big(-\partial_2(-\Delta)^{-\beta/2}, \partial_1(-\Delta)^{-\beta/2}\Big) = \nabla^\perp(-\Delta)^{-\beta/2}, \tag{3.3}$$

which are bounded in the $H^s \longrightarrow H^{s+\beta-1}$ topology. The operators A_1 and A_2 are formally skew-symmetric, as a consequence of Plancherel's identity: set $im(\xi) = i\xi^\perp/|\xi|^\beta \in i\mathbb{R}$ so that, for any $f, g \in \mathcal{S}$ and $j \in \{1, 2\}$, we have

$$\begin{aligned}\int f A_j g \,\mathrm{d}x &= C \int \widehat{f}(\xi)\overline{im_j(\xi)\widehat{g}(\xi)}\,\mathrm{d}\xi \\ &= -C \int im_j(\xi)\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,\mathrm{d}\xi \\ &= -\int A_j f g \,\mathrm{d}x.\end{aligned}$$

This is only a generalization of the usual integration by parts, by using Fourier multipliers with pure imaginary symbols. As a consequence, when we multiply the equation (3.2) by θ and integrate, we have (with an implicit summation on the repeated index $j = 1, 2$)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\theta| \,\mathrm{d}x &= -\int \theta.\partial_j\theta.A_j\theta \,\mathrm{d}x \\ &= \int A_j(\partial_j\theta.\theta).\theta \,\mathrm{d}x \\ &= \frac{1}{2}\int \theta.\Big(A_j(\partial_j\theta.\theta) - \partial_j\theta.A_j\theta\Big)\,\mathrm{d}x.\end{aligned}$$

By introducing the multiplication operator $\partial_j\theta : f \longmapsto \partial_j\theta.f$, we see that the difference in the equation above takes the form of a commutator of operators

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\theta| \,\mathrm{d}x = \frac{1}{2}\int \theta.\big[A_j, \partial_j\theta\big]\theta \,\mathrm{d}x.$$

Therefore, estimating the L^2 norm of the solution can be done by finding a L^2 estimate for a commutator of operators. While this may seem to be a convoluted route to find a basic estimate, we will see that this general commutator structure, put to use in high order estimates, is the key to well-posedness for SQG.

3.3 A Commutator Estimate

This paragraph focuses on finding L^2 bounds for commutators of the form $[A, g]f$, where A is the Fourier multiplier from (3.3) and g is a given function. The general philosophy of such commutators is as follows. Naively, the operator $[A, g]$ ought to be of order $1 - \beta$. However, is is possible to show that, provided g is regular enough, it is possible to "absorb" one derivative in g, and show that the operator is in fact of order $-\beta$. In a way, this is "trading" the regularity of g for a lower order. There is an immense number of such commutator estimates in the mathematical literature using techniques that range from simple Fourier transform manipulations (as below) or mean value inequalities, to highly technical methods based on maximal functions, Littlewood-Paley inequalities or paradifferential calculus (the Kato-Ponce estimates for example, or the different commutator lemmas in Section 2.10 of [2]).

Proposition 37 (Proposition 2.1 in [7]). *Consider $\beta \in \mathbb{R}$ and the operator $A = \nabla^\perp(-\Delta)^{-\beta/2}$. For any $\epsilon > 0$ the bound*

$$\left\|[A, g]f\right\|_{L^2} \le C(\beta)\|f\|_{L^2} \left\|\widehat{(-\Delta)^{(1-\beta)/2}g}\right\|_{L^1} + C(\beta)\|(-\Delta)^{-\beta/2}f\|_{L^2} \left\|\widehat{(-\Delta)^{1/2}g}\right\|_{L^1}.$$

holds as long as all the norms above make sense (and are finite).

Remark 38. Usually, this inequality is stated in the following way: assuming that $\beta \in \mathbb{R}$, the commutator $[A, g]f$ satisfies the bound

$$\left\|[A, g]f\right\|_{L^2} \le C(\beta, \epsilon)\Big(\|f\|_{L^2}\|g\|_{\dot{H}^{2-\beta+\epsilon}} + \|f\|_{\dot{H}^{-\beta}}\|g\|_{\dot{H}^{2+\epsilon}}\Big).$$

In the above, the space $\dot{H}^s$ is the *homogeneous* Sobolev space of distributions $f \in \mathcal{S}'$ such that the Fourier transform $\widehat{f}$ belongs to the space $L^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ and[3] the integral

$$\|f\|^2_{\dot{H}^s} := \int |\xi|^{2s}|\widehat{f}(\xi)|^2 \mathrm{d}\xi$$

finite. Note that in general $\| \,.\, \|_{\dot{H}^s}$ does not define a norm but only a semi-norm, as some nonzero functions may have a zero norm. For example, the constant function $f = 1$ has Fourier transform $\widehat{f} = \delta_0$, which belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$, and cancels the norm $\|f\|_{\dot{H}^s}$. This means that homogeneous Sobolev spaces are very tricky to deal with!

Proof. We start by writing the Fourier transform of the commutator $[A, g]f$. Since function products translate as convolution products, we have

$$\mathcal{F}\big([A, g]f\big)(\xi) = \int \Big(|\xi|^{-\beta}\xi - |\xi-\eta|^{-\beta}(\xi-\beta)\Big)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\mathrm{d}\eta.$$

We will estimate the rational function in this integral by a mean value inequality. By setting

$$B(\tau) = \tau\xi + (1-\tau)(\xi-\eta) \qquad \text{for } \tau \in [0, 1],$$

so that $B(0) = \xi - \eta$ and $B(1) = \xi$, we see that

$$\begin{aligned}
|\xi|^{-\beta}\xi - |\xi-\eta|^{-\beta}(\xi-\beta) &= |B(1)|^{-\beta}B(1) - |B(0)|^{-\beta}B(0) \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\big(|B(\tau)|^{-\beta}B(\tau)\big)\mathrm{d}\tau \\
&= \int_0^1 \Big(\eta|B(\tau)|^{-\beta} - \beta|B(\tau)|^{-\beta-2}(B(\tau)\cdot\eta)B(\tau)\Big)\mathrm{d}\tau.
\end{aligned} \tag{3.4}$$

Now, let us first consider the case where $\beta \le 0$. A triangular inequality applied to (3.4) provides the inequality

$$\begin{aligned}
\Big||\xi|^{-\beta}\xi - |\xi-\eta|^{-\beta}(\xi-\beta)\Big| &\le (1+|\beta|)|\eta|\int_0^1 |B(\tau)|^{-\beta}\mathrm{d}\tau \\
&\le (1+|\beta|)|\eta| \max\left\{|\xi|^{-\beta}, |\xi-\eta|^{-\beta}\right\},
\end{aligned}$$

which we may directly apply to the commutator and obtain a Fourier transform bound.

$$\big|\mathcal{F}\big([A, g]f\big)(\xi)\big| \le C(\beta)|\xi|^{-\beta}\int |\eta|\big|\widehat{g}(\eta)\big|\,\big|\widehat{f}(\xi-\eta)\big|\mathrm{d}\eta + C(\beta)\int |\eta|\big|\widehat{g}(\eta)\big|\,|\xi-\eta|^{-\beta}\big|\widehat{f}(\xi-\eta)\big|\mathrm{d}\eta$$

[3]This means that $\widehat{f}$ is square-integrable on every compact subset of $\mathbb{R}^d \setminus \{0\}$, without assuming anything on the behavior of $\widehat{f}$ near $\xi = 0$.

Moreover, we make use of the fact that the norms $|\,.\,|_1$ and $|\,.\,|_\infty$ are equivalent in finite dimension to write the basic inequality

$$|\xi|^{-\beta} \leq \Big(|\xi-\eta| + |\eta|\Big)^{-\beta} \leq \Big(2\max\big\{|\xi-\eta|, |\eta|\big\}\Big)^{-\beta} \leq 2^{|\beta|}\Big(|\xi-\eta|^{-\beta} + |\eta|^{-\beta}\Big),$$

from which we deduce that

$$\big|\mathcal{F}\big([A,g]f\big)(\xi)\big| \leq C(\beta)\int |\eta|^{1-\beta}|\widehat{g}(\eta)|\big|\widehat{f}(\xi-\eta)\big|\mathrm{d}\eta + C(\beta)\int |\eta||\widehat{g}(\eta)||\xi-\eta|^{-\beta}\big|\widehat{f}(\xi-\eta)\big|\mathrm{d}\eta.$$

Applying the Hausdorff-Young convolution inequality along with the Plancherel identity to the above provides an L^2 bound on the commutator, namely

$$\big\|\mathcal{F}\big([A,g]f\big)(\xi)\big\|_{L^2} \leq C(\beta)\|f\|_{L^2}\left\|\widehat{(-\Delta)^{(1-\beta)/2}g}\right\|_{L^1} + C(\beta)\big\|(-\Delta)^{-\beta/2}f\big\|_{L^2}\left\|\widehat{(-\Delta)^{1/2}g}\right\|_{L^1},$$

which is the inequality we were seeking.

We now take care of the case where $\beta > 0$. By invoking Fubini's theorem and equation (3.4), we may write the Fourier transform of the convolution as a double integral on (τ,η).

$$\begin{aligned}\mathcal{F}\big([A,g]f\big)(\xi) = \int_0^1\int \eta|B(\tau)|^{-\beta}\widehat{g}(\eta)\widehat{f}(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau \\ - \beta\int_0^t\int |B(\tau)|^{-\beta-2}(B(\tau)\cdot\eta)B(\tau)\widehat{g}(\eta)\widehat{f}(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau.\end{aligned}$$

In particular, we have a common upper bound for both a)of the above integrals:

$$\big|\mathcal{F}\big([A,g]f\big)(\xi)\big| \leq C(\beta)\int_0^1\int |B(\tau)|^{-\beta}|\eta||\widehat{g}(\eta)|\big|\widehat{f}(\xi-\eta)\big|\,\mathrm{d}\eta\,\mathrm{d}\tau.$$

Because we are in the case where $\beta > 0$, the function $x \mapsto |x|^{-\beta}$ is convex. Using this on $B(\tau)$, which we reformulate as a convex combination of η and $\xi-\eta$, we obtain

$$\begin{aligned}|B(\tau)|^{-\beta} &= \Big|\tau\xi + (1-\tau)(\xi-\eta)\Big|^{-\beta} \\ &= \Big|\xi-\eta+\tau\eta\Big|^{-\beta} \\ &= (1+\tau)^{-\beta}\left|\frac{1}{1+\tau}(\xi-\eta) + \frac{\tau}{1+\tau}\eta\right|^{-\beta} \\ &\leq (1+\tau)^{-\beta-1}|\xi-\eta|^{-\beta} + \tau(1+\tau)^{-\beta-1}|\eta|^{-\beta}.\end{aligned}$$

Plugging this in the integral inequality yields the bound

$$\begin{aligned}\big|\mathcal{F}\big([A,g]f\big)(\xi)\big| \leq C(\beta)\left(\int_0^1(1+\tau)^{-\beta-1}\mathrm{d}\tau\right)\int |\eta||\widehat{g}(\eta)||\xi-\eta|^{-\beta}\big|\widehat{f}(\xi-\eta)\big|\,\mathrm{d}\eta \\ + C(\beta)\left(\int_0^1\tau(1+\tau)^{-1-\beta}\mathrm{d}\tau\right)\int |\eta|^{1-\beta}|\widehat{g}(\eta)|\big|\widehat{f}(\xi-\eta)\big|\,\mathrm{d}\eta\end{aligned}$$

Both τ-integrals are simply continuous functions of $\beta > 0$. We may therefore apply the Hausdorff-Young convolution inequality in order to obtain an inequality identical to (3.3), albeit with different constants. As it holds whatever the value of β, we have finished the proof. $\square$

3.4 Local Well-Posedness of SQG

In this long paragraph, we will do the same work we did for the Euler equations, and show the existence and uniqueness of local regular solutions for the SQG system. Once again, the result we present is not optimal, and we will shortly comment on how it has been improved in subsequent studies.

Theorem 39 (Chae, P. Constantin, Córdoba, Gandeco, Wu, 2012, [7])**.** *Consider an exponent $\beta \in]0, 2[$ and an integer $k \geq 4$. Then, for any initial datum $\theta_0 \in H^k$, there exists a time $T > 0$ such that the SQG system has a unique local solution $\theta \in L^\infty_T(H^k)$ associated to the initial datum θ_0. Moreover, the time T can be taken such that*

$$T \geq \frac{C(\beta, k)}{\|\theta_0\|_{H^k}}.$$

The fact that Theorem 39 is (relatively) recent, and yet can be included is this (hopefully) accessible introduction to fluid dynamics shows that not all modern research in PDEs is about using the most sophisticated tools from functional or harmonic analysis. A very large part of the work lies in understanding the *structure* of a PDE, in our case the apparition of a commutator: Theorem 39 is very impressive because one would not have guessed at first glance that SQG is well-posed, yet, as we will see it is strikingly elementary.

However, the elementary aspect of the theorem means that it is not optimal (we will come back to this point). This can already seen from the assumption on the initial datum, which is required to be at least H^4. However, a more natural regularity hypotheses on θ_0 should be dependent on the value of β, for example $\theta_0 \in H^s$ with $s > 3 - \beta$.

It turns out that such improvements have been made in [22] and [20]. But the fact that the regularity exponents used in these papers are non-integral, unlike $k \geq 4$ in Theorem 39, makes it necessary to resort to much more advanced tools. The reason for that is quite simple: it is much easier to estimate the H^k norm of a product of functions fg (with the Leibnitz rule) than it is to estimate its H^s norm. And this boils down to the fact that there is no fractional Leibniz rule (although there are analogous product decompositions).

As a final remark, the fact that Theorem 39 belongs to the landscape of modern research means that there still might be improvements within the reach of mathematicians, For example, by improving the condition on the initial datum.

3.4.1 *A Priori* Estimates

As for the Euler equations, and this is the norm in PDEs, the first step of the proof is to make sure, through formal computations, that a H^k initial data will "naturally" yield H^k solutions, at least for some time.

Consider a smooth and sufficiently integrable solution θ of the SQG equation associated with a smooth and sufficiently integrable initial datum θ_0. Our goal is to bound the norm $\|\theta(t)\|_{H^k}$ by a function of $\|\theta_0\|_{H^k}$, on some time interval $[0, T[$. Just as with the Euler equations, we will use energy estimates. Let $\alpha \in \mathbb{N}^2$ be a multi-index of length $|\alpha| = k$. Then applying ∂^α to the first equation in (3.1), multiplying by $\partial^\alpha \theta$ and integrating gives

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int |\partial^\alpha \theta|^2 \,\mathrm{d}x + \int \partial^\alpha \theta . \partial^\alpha \operatorname{div}\big(A\theta . \theta\big) \,\mathrm{d}x = 0. \tag{3.5}$$

Recall that, because $A\theta$ is divergence-free, we have the identity $\operatorname{div}(A\theta . f) = A\theta \cdot \nabla f$. By using the "improved" binomial formula, as in (2.10), when we were looking for *a priori* estimates for

Euler's equation, we obtain, in the same way

$$\begin{aligned}\int \partial^\alpha\theta.\partial^\alpha \operatorname{div}\big(A\theta.\theta\big)\,\mathrm{d}x &= \sum_{|\gamma|\le k}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\int \partial^\alpha\theta.\partial^\gamma\nabla\theta\cdot\partial^{\alpha-\gamma}A\theta\,\mathrm{d}x\\ &= \sum_{|\gamma|\le k}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix} I_\gamma.\end{aligned}$$

We distinguish between four cases when trying to evaluate the integrals I_γ.

First case. We start by looking at the case where $2 \le |\gamma| \le k-1$. Then, we have, by using the embeddings of Lemma 32, $H^{1+\epsilon} \subset L^\infty$ for any $\epsilon > 0$, and

$$\begin{aligned}|I_\gamma| &\le \|\nabla^k\theta\|_{L^2}\|\nabla^{|\gamma|+1}\theta\|_{L^2}\|\nabla^{k-|\gamma|}A\theta\|_{L^\infty}\\ &\le C(\epsilon)\|\theta\|_{H^k}^2\|\nabla^{k-|\gamma|}A\theta\|_{H^{1+\epsilon}}.\end{aligned}$$

Recall that the operator A is of degree $1-\beta<1$, so that, by choosing ϵ satisfying $1-\beta+\epsilon<1$, we have

$$|I_\gamma| \le C(\beta)\|\theta\|_{H^k}^2\|\theta\|_{H^{k-2-|\gamma|-\beta+\epsilon}} \le C(d)\|\theta\|_{H^k}^3.$$

Second case. We now study the case where $|\gamma| = 1$. The computations are quite similar: by letting the L^∞ norm fall on $\partial^\gamma\nabla\theta$ in stead, we have

$$\begin{aligned}|I_\gamma| &\le \|\nabla^\alpha\theta\|_{L^2}\|\nabla^2\theta\|_{L^\infty}\|\nabla^{k-1}A\theta\|_{L^2}\\ &\le C(\epsilon)\|\theta\|_{H^k}^2\|\theta\|_{H^{3+\epsilon}}\\ &\le C(\epsilon)\|\theta\|_{H^k}^3.\end{aligned}$$

Because k is an integer, we see that we must take $k\ge 4$ for the last inequality to hold.

Third case. We look at the case where $\gamma=\alpha$. Here, we use the fact that the vector field $u = A\theta$ is divergence-free, since it is the gradient-orthogonal of a scalar field and $\operatorname{div}(\nabla^\perp)=0$. This implies that a simple integration by parts shows that the integral I_α is zero:

$$\begin{aligned}I_\alpha &= \int \partial^\alpha\theta.\operatorname{div}(A\theta.\partial^\alpha\theta)\,\mathrm{d}x\\ &= -\int \nabla\partial^\alpha\theta\cdot A\theta.\partial^\alpha\theta\,\mathrm{d}x\\ &= -\int \operatorname{div}(\partial^\alpha\theta.A\theta).\partial^\alpha\theta\,\mathrm{d}x\\ &= -I_\alpha = 0.\end{aligned}$$

Fourth case. Finally, we study the case where $\gamma=0$. This is the hardest case, because the quantity $\|\partial^\alpha A\theta\|_{L^2}$ cannot be bounded in terms of the H^k norm of θ, at least when $\beta<1$, so the operator A is of positive order $1-\beta>0$. It is at this point that we resort to the commutator structure of the SQG equation. By recalling that A_j is formally skew-symmetric, we have

$$\begin{aligned}I_0 &= \int \partial^\alpha\theta.\partial_j\theta.\partial^\alpha A_j\theta\,\mathrm{d}x\\ &= -\int A_j\big(\partial_j\theta.\partial^\alpha\theta\big).\partial^\alpha\theta\,\mathrm{d}x\\ &= \frac{1}{2}\int \partial^\alpha\theta.\big[\partial_j\theta, A_j\big]\partial^\alpha\theta\,\mathrm{d}x.\end{aligned}$$

We apply our commutator estimate Proposition 37 to the above. We obtain the inequality

$$
\begin{aligned}
|I_0| &\leq C(\beta)\|\partial^\alpha\|_{L^2}\sum_j\left(\|\partial^\alpha\theta\|_{L^2}\left\|\widehat{(-\Delta)^{(1-\beta)/2}\partial_j\theta}\right\|_{L^1}+\left\|(-\Delta)^{-\beta/2}\partial^\alpha\theta\right\|_{L^2}\left\|\widehat{(-\Delta)^{1/2}\partial_j\theta}\right\|_{L^1}\right)\\
&\leq C(\beta)\|\theta\|_{H^k}\sum_j\Bigg[\|\theta\|_{H^k}\int|\xi|^{2-\beta}|\widehat{\theta}(\xi)|\mathrm{d}\xi\\
&\qquad\qquad+\left(\int\left||\xi|^{k-\beta}\widehat{\theta}(\xi)\right|^2\mathrm{d}\xi\right)^{1/2}\left(\int|\xi|^2|\widehat{\theta}(\xi)|\mathrm{d}\xi\right)\Bigg].
\end{aligned}
$$

We have to check that all the integrals above can be bounded by $\|\theta\|_{H^k}$. First of all, note that the functions $|\xi|^{2-\beta}$, $|\xi|^{k-\beta}$ and $|\xi|^2$ are all bounded on the unit ball $|\xi|\leq 1$, as $\beta>0$ and $k\geq 4$ by assumption. We deduce that

$$
\left(\int\left||\xi|^{k-\beta}\widehat{\theta}(\xi)\right|^2\mathrm{d}\xi\right)^{1/2}\leq\|\theta\|_{H^k}\tag{3.6}
$$

and that, for any $\epsilon>0$,

$$
\begin{aligned}
\int|\xi|^{2-\beta}|\widehat{\theta}(\xi)|\mathrm{d}\xi+\int|\xi|^2|\widehat{\theta}(\xi)|\mathrm{d}\xi &\leq\int\langle\xi\rangle^2|\widehat{\theta}(\xi)|\mathrm{d}\xi\\
&\leq\int\langle\xi\rangle^{3+\epsilon}|\widehat{\theta}(\xi)|\frac{\mathrm{d}\xi}{\langle\xi\rangle^{1+\epsilon}}\\
&\leq C(\epsilon)\|\theta\|_{H^{3+\epsilon}}.
\end{aligned}
$$

Bu choosing ϵ so that $1+\epsilon\leq 4$, the norm above can be bounded by $\|\theta\|_{H^4}$ (once again, we see the necessity of taking $k\geq 4$). Overall, we have obtained the inequality

$$
|I_0|\leq C(\beta)\|\theta\|_{H^k}^3.
$$

We have just bounded the norms $\|\partial^\alpha\theta\|_{L^2}$ by $\|\theta\|_{H^k}^3$, and in doing so, we have used the fact that $|\alpha|=k\geq 4$. This was crucial in bounding the commutator, otherwise the factor $|\xi|^{k-\beta}$ appearing in (3.6) would not be bounded on the unit ball $|\xi|\leq 4$. However, this means that we have not yet bounded the full H^k norm:

$$
\|\theta\|_{H^k}\approx\|\theta\|_{L^2}+\sum_{|\alpha|=k}\|\partial^\alpha\theta\|_{L^2}.
$$

We still are missing an estimate for the norm $\|\theta\|_{L^2}$. Thankfully, it is easy to obtain, since the velocity field $u=A\theta$ is divergence-free $\operatorname{div}(u)=0$. We can use integration by parts to write

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\theta|^2\,\mathrm{d}x &=-\int\theta\operatorname{div}(\theta.A\theta)\,\mathrm{d}x\\
&=\int\nabla\theta\cdot A\theta.\theta\,\mathrm{d}x\\
&=\int\theta\operatorname{div}(\theta.A\theta)\,\mathrm{d}x=0.
\end{aligned}
$$

In other words, the L^2 norm of the unknown is conserved: $\|\theta(t)\|_{L^2}=\|\theta_0\|_{L^2}$.

Putting everything together, we have proven that the solution θ solves a differential inequality of the same form as the one we had for the Euler equation.

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\theta\|_{H^k}^2\right)\leq C(\beta)\|\theta\|_{H^k}^3.
$$

By exactly reproducing computations we performed with the Euler equation, it is possible to obtain bounds on $\|\theta\|_{H^k}$ on a finite time interval. In order to accommodate the reader, we remind the main lines of the argument. First of all, by integrating the differential inequality above, we have

$$\|\theta(t)\|_{H^k} \leq \|\theta_0\|_{H^k} + C(\beta) \int_0^t \|\theta(s)\|_{H^k}^2 \, \mathrm{d}s.$$

We then define a time $T > 0$ by setting

$$T = \sup \left\{ t \geq 0, \quad C(\beta) \int_0^t \|\theta(s)\|_{H^k}^2 \, \mathrm{d}s \leq \|\theta_0\|_{H^k} \right\}.$$

On the time interval, we have the inequality $\|\theta(t)\|_{H^k} \leq 2\|\theta_0\|_{H^k}$, from which it follows that T must be at least $T \geq C(\beta)/\|\theta_0\|_{H^k}$. This completes the quest for $L^\infty_T(H^k)$ *a priori* bounds.

3.4.2 Construction of a Solution

As was the case for the Euler equations, we will construct a solution of the SQG system (3.1) by taking a limit of solutions of a, better behaved, approximate system. As we will see, the proof is quite comparable to the one for the Euler equations. Once again, we take the Fourier projection operator P_n defined by

$$\forall f \in L^2, \qquad \widehat{P_n f}(\xi) = \mathbb{1}_{|\xi| \leq n} \widehat{f}(\xi)$$

and study the approximate system

$$\partial_t \theta_n + P_n \operatorname{div}(\theta_n . A\theta_n) = 0 \tag{3.7}$$

to which we associate the initial datum $\theta_n(0) = P_n \theta_0$.

STEP 1: approximate solutions. As in the case of the Euler equations, we wish to apply the Cauchy-Lipschitz theorem to show the existence of solutions for the approximate system (3.7). We define the Banach space

$$X_n := \Big\{ f \in L^2, \quad \operatorname{supp}(\widehat{f}) \subset B(0, n) \Big\},$$

which is the subspace of L^2 on which P_n is an orthogonal projection. All functions of X_n are C^∞ smooth, and we have, for any $f \in X_n$,

$$\|f\|_{L^\infty} \leq C\|f\|_{H^{1+\epsilon}} \leq Cn^{1+\epsilon}\|f\|_{L^2}.$$

We have to check that the map $\theta \mapsto P_n \operatorname{div}(\theta . A\theta)$ is locally Lipschitz over X_n. Let us fix two functions $f, g \in X_n$. We see that

$$\begin{aligned} \big\| P_n \operatorname{div}(f.Af) - \operatorname{div}(g.Ag) \big\|_{L^2} &\leq n \|f.Af - g.Ag\|_{L^2} \\ &\leq n\|f - g\|_{L^2} \|Af\|_{L^\infty} + \|g\|_{L^\infty} \|A(f-g)\|_{L^2}. \end{aligned} \tag{3.8}$$

Because f and g are both elements of X_n, and because A is a Fourier multiplier, we have

$$\|Af\|_{L^\infty} \leq C\|Af\|_{H^{1+\epsilon}} \leq Cn^{2-\beta+\epsilon}\|f\|_{L^2},$$

and likewise for Ag. This means that inequality (3.8) becomes

$$\big\| P_n \operatorname{div}(f.Af) - \operatorname{div}(g.Ag) \big\|_{L^2} \leq Cn^{2-\beta+\epsilon} \|f - g\|_{L^2} \Big(\|f\|_{L^2} + \|g\|_{L^2} \Big),$$

which proved that the map we were considering is locally Lipschitz in X_n. The Cauchy-Lipschitz theorem therefore shows the existence of a (unique) maximal solution $\theta_n \in C^0([0, T_n[; X_n)$ associated to the initial datum $P_n \theta_0$.

STEP 2: approximate solutions are global. The lifespan T_n of the approximate solutions is infinite $T_n = +\infty$. For this, we show that the ODE solution θ_n remains bounded in the space X_n. By multiplying the equation (3.7) by θ_n and integrating, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\theta_n|^2 \,\mathrm{d}x = -\int \theta_n.P_n \operatorname{div}(\theta_n.A\theta_n)\,\mathrm{d}x.$$

Because $P_n : L^2 \longrightarrow X_n$ is an orthogonal projection, the P_n is not needed in the expression above. Now, recall that the vector field $u_n = A\theta_n$ is divergence free $\operatorname{div}(A\theta_n) = 0$. This means that we can use integration by parts to show that the righthand side integral is equal to its opposite, and is therefore zero:

$$\begin{aligned} -\int \theta_n.\operatorname{div}(\theta_n.A\theta_n)\,\mathrm{d}x &= -\int \theta_n.\nabla\theta_n \cdot A\theta_n \,\mathrm{d}x \\ &= \int \operatorname{div}(\theta_n.A\theta_n).\theta_n \,\mathrm{d}x = 0. \end{aligned}$$

We deduce that the L^2 norms of the approximate solutions are preserved, so that they remain bounded in X_n. As in the case of the Euler equations, this implies that the solutions are global $T_n = +\infty$. by Lemma 22.

STEP 3: compactness of the Approximate Solutions. First of all, we should note that the *a priori* estimates we have proved above also hold for the approximate solutions. Indeed, applying the derivative ∂^α to (3.7), multiplying by $\partial^\alpha\theta_n$ and integrating gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial^\alpha\theta_n|^2 \,\mathrm{d}x + \int \partial^\alpha\theta_n.\partial^\alpha P_n \operatorname{div}(A\theta.\theta)\,\mathrm{d}x = 0.$$

Because $\theta_n \in X_n$, we also have $\partial^\alpha\theta_n \in X_n$, and so the Fourier projection P_n can be omitted in the formula above, thus reducing it to equation (3.5). By performing exactly the same steps as in the *a priori* estimates, we see that we must have

$$\sup_{[0,T_n]} \|\theta_n\|_{H^k} \leq 2\|P_n\theta_0\|_{H^k} \leq 2\|\theta_0\|_{H^k},$$

where the times T_n fulfill the inequality

$$T_n \geq \frac{C(k)}{\|A_n\theta_0\|_{H^k}} \geq \frac{C(k)}{\|\theta_0\|_{H^k}} := T.$$

We deduce that the sequence (θ_n) is bounded in the space $L^\infty_T(H^k)$. By using the Rellich-Kondrachov theorem, this should provide compactness with respect to the space variable. But to use Ascoli's theorem (Theorem 24), we also need uniform equicontinuity with respect to the time variable. This we find by looking at the time derivative $\partial_t\theta_n$. Because θ_n is a solution of the approximate system, we have

$$\begin{aligned} \|\partial_t\theta_n\|_{H^2} &= \big\|P_n \operatorname{div}(\theta_n.A\theta_n)\big\|_{H^2} \\ &\leq \|\theta_n.A\theta_n\|_{H^3}. \end{aligned}$$

We are seeking H^3 bounds on the product $\theta_n.A\theta_n$. Because the Sobolev norm is given by $\|f\|_{H^3} \approx \|f\|_{L^2} + \|\nabla^3 f\|_{L^3}$, this means that we have two quantities to estimate. On the one hand, thanks to Lemma 36, we have

$$\|\theta_n.A\theta_n\|_{L^2} \leq \|\theta_n\|_{L^2}\|A\theta_n\|_{L^\infty} \leq C\|\theta_n\|^2_{H^3}.$$

On the other hand, we look at the third order derivative $\|\nabla^3(\theta_n.A\theta_n)\|_{L^2}$. By using an improved binomial formula, the Sobolev embeddings from Lemma 32 and Lemma 36, we write

$$\begin{aligned}\|\nabla^3(\theta_n A\theta_n)\|_{L^2} &\leq \|\theta_n\|_{L^\infty}\|\nabla^3 A\theta_n\|_{L^2} + 3\|\nabla\theta_n\|_{L^\infty}\|\nabla^2 A\theta_n\|_{L^2} + 3\|\nabla^2\theta_n\|_{L^2}\|\nabla A\theta_n\|_{L^\infty}\\ &\qquad + \|A\theta_n\|_{L^\infty}\|\nabla^3\theta_n\|_{L^2}\\ &\leq C\|\theta_n\|_{H^3}^2.\end{aligned}$$

From Proposition 35, we know that the sequence $(A\theta_n)$ is bounded in the space $L^\infty_T(H^{k-1}) \subset L^\infty_T(H^3)$. Therefore,

$$\begin{aligned}|\partial_t\theta_n\|_{H^2} &\leq \|\theta_n.A\theta_n\|_{H^3} \leq C\|\theta_n\|_{H^3}\|A\theta_n\|_{H^3}\\ &\leq C\|\theta_n\|_{H^4}^2 \leq C\|\theta_0\|_{H^k}^2.\end{aligned}$$

This means that the sequence (θ_n) is additionally bounded in the space $W^{1,\infty}_T(H^2)$. Applying the Rellich-Kondrachov and Ascoli theorems, we obtain strong convergence of the approximate solutions

$$\theta_n \longrightarrow \theta \qquad \text{in } C^0_T(H^1_{\text{loc}}). \tag{3.9}$$

Let us now focus on the convergence of the velocity fields $u_n = A\theta_n$. We cannot immediately deduce the convergence $A\theta_n \to A\theta$ from the convergence (3.9) above, which is local (*i.e.* in the space H^1_{loc}), whereas the operator A is non-local. Instead, we will resort to a weak type of convergence: note that thanks to the bounds in Lemma 36 the sequence (v_n) is bounded in the space $L^\infty_T(L^\infty)$ (in fact, it is bounded in $L^\infty_T(W^{1,\infty})$ but it will not be useful). In particular, the Banach-Alaoglu theorem provides the weak-$(*)$ convergence of an extraction (which we omit)

$$A\theta_n \overset{*}{\rightharpoonup} v \qquad \text{in } L^\infty_T(L^\infty),$$

for some function $v \in L^\infty_T(L^\infty)$. Let us prove that we do indeed have $v = A\theta$. We fix a test function $\phi(t,\xi) \in \mathcal{D}([0,T[\times\mathbb{R}^d)$ and we see that $i\xi^\perp|\xi|^{-\beta}\widehat{\phi}(t,\xi) \in L^2_T(L^2)$. Furthermore, since we have the weak convergence $\widehat{\theta} \rightharpoonup^* \widehat{\theta}$ in $L^\infty_T(L^2)$, we find that

$$\begin{aligned}\int_0^T\int \widehat{A\theta_n}(t,\xi)\cdot\widehat{\phi}\,\mathrm{d}\xi\,\mathrm{d}t &= \int_0^T\int \widehat{\theta_n}(t,\xi)\frac{i\xi^\perp}{|\xi|^\beta}\cdot\phi(t,\xi)\mathrm{d}\xi\,\mathrm{d}t\\ &\longrightarrow \int_0^T\int \widehat{\theta}(t,\xi)\frac{i\xi^\perp}{|\xi|^\beta}\cdot\phi(t,\xi)\mathrm{d}\xi\,\mathrm{d}t\\ &= \int_0^T\int \widehat{A\theta}\cdot\widehat{\phi}\,\mathrm{d}\xi\,\mathrm{d}t.\end{aligned}$$

The weak convergence $A\theta_n \rightharpoonup v$ in $L^\infty_T(L^\infty)$ implies that we must in fact have $v = A\theta$.

STEP 4: the limit is a solution. We know that the approximate solutions (θ_n) solve the systems

$$\begin{cases}\partial_t\theta_n + P_n\,\mathrm{div}(\theta_n.A\theta_n) = 0\\ \theta_n(0) = P_n\theta_0,\end{cases}$$

and we wish to let $n \to +\infty$ to show that the function θ is a solution of the SQG equations. Firstly, if $\phi \in \mathcal{D}(]0,T[\times\mathbb{R}^d)$, it is immediate that

$$\begin{aligned}\langle\partial_t\theta_n,\phi\rangle &= -\int_0^T\int \theta_n\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t\\ &\longrightarrow -\int_0^T\int \theta\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t = \langle\partial_t\theta,\phi\rangle,\end{aligned}$$

so that we have the convergence $\partial_t\theta_n \longrightarrow \partial_t\theta$ in the sense of distributions $\mathcal{D}'(]0,T[\times\mathbb{R}^d)$. Next, we look at the convergence of the product term $\theta_n.A\theta_n$.

[WORK IN PROGRESS]

Chapter 4

The Navier-Stokes Equations

The Navier-Stokes (NS for short) equations bear a formal resemblance to the Euler equations. They are identical up to the addition of a term representing the effect of viscosity:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla \pi = \nu \Delta u \\ \operatorname{div}(u) = 0. \end{cases} \tag{4.1}$$

In the above, $\nu > 0$ is the viscosity coefficient. For example, honey is a highly viscous fluid for which the NS equations are a much superior model than the Euler equations: the inviscid approximation simply is not reasonable on any time scale.

In this chapter, we will consider only the the case of Newtonian fluids: the viscosity $\nu > 0$ will be taken constant. In fact, as the precise value of ν has no mathematical relevance for the material in this chapter, we will even assume that $\nu = 1$ to simplify computations.

Because of the proximity of the Navier-Stokes and Euler equations, many of techniques we used above will also be useful in this chapter. For instance, we will resort to the Leray projection operator $\mathbb{P} = \mathrm{Id} + \nabla(-\Delta)^{-1} \operatorname{div}$ in order to eliminate the pressure from the equations, which become

$$\partial_t u - \Delta u = -\mathbb{P}\big((u \cdot \nabla)u\big).$$

However, there is a fundamental difference between the Navier-Stokes and Euler systems. When there is no viscosity, we have seen that the solution has no reason to be more regular than the initial datum: initial regularity is propagated by the evolution. Here, things will be much different, as NS is basically a heat equation with an added non-linear non-local term, and just as in the case of the heat equation, we should expect solutions to have better regularity than the initial datum.

The Navier-Stokes equations are famous for being the object of one of the millennium problems of the Clay institute, with a million US dollar prize attached to it. The question is the following: given an initial datum $u_0 \in \mathcal{S}$, is it possible to prove or disprove the existence (and uniqueness) of a global in time, finite energy, smooth solution for the 3D Navier-Stokes equations? (the 2D problem is, as we will see, much easier) Although considerable progress has been made since the problem was first studied, some 90 years ago [23], it still resists the efforts of mathematicians. While we cannot give here a complete panorama of a near century of research, we will cover some recent results in this chapter.

Warning! Before we move on, a word of caution is in order. For time constraints, this chapter will have a number of somewhat formal computations. We try to tell the reader whenever this happens. However, upgrading the arguments we present to actual proofs should not be too difficult, given the background the first two chapters has set.

Remark 40. It should be noted that many fluids do not have a constant viscosity coefficient: in general $\nu \Delta u$ can be replaced by a nonlinear expression. A typical example is the p-Laplacian Δ_p

defined by

$$\nu\Delta_p u := \nu\,\mathrm{div}(|Du|^{p-2}Du),$$

where $Du = \frac{1}{2}(\nabla u + {}^t\nabla u)$ is the deformation tensor of the fluid. A non-linear viscosity adds considerable mathematical difficulties in the analysis, which goes far beyond the scope of these lectures.

4.1 Energy Estimates and Weak Leray Solutions

As we have said above, the added viscosity has a regularizing effect on the solutions. This can already be seen through a simple energy estimate: to evaluate the evolution of the kinetic energy in the fluid, we take the scalar product of the first equation in (4.1) and integrate with respect to space. We obtain (assuming the solution u is nice enough)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\,\mathrm{d}x + \int (u\cdot\nabla)u\cdot u\,\mathrm{d}x + \int \nabla\pi\cdot u\,\mathrm{d}x = \int \Delta u\cdot u\,\mathrm{d}x.$$

In the above, the second and third integrals are zero. For the second one, integration by parts shows that it is equal to its opposite (and so it is zero), while integration by parts in the pressure integral and the divergence condition $\mathrm{div}(u) = 0$ gives the cancellation. Furthermore, by using the fact that the Laplace operator can be written as a composition of a gradient and a divergence $\Delta = \mathrm{div}(\nabla)$, we may also integrate by parts in the last integral to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\,\mathrm{d}x + \int |\nabla u|^2\,\mathrm{d}x = 0,$$

or, after integration on a time interval $[0,T]$,

$$\frac{1}{2}\|u(T)\|_{L^2}^2 + \int_0^T\int |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\|u_0\|_{L^2}^2. \tag{4.2}$$

This already provides an important lesson about the Navier-Stokes equations: the kinetic energy decreases with time, and whatever energy is lost by the fluid is transformed into heat by viscosity effect. The space-time integral above is simply the total amount of energy lost through the process of viscosity. In particular, the dynamics of a viscous fluid are irreversible: once time has passed, it is impossible to revert to the initial state, as the lost energy would have to be regained.

Mathematically speaking, the energy balance equation (4.2) also shows that the natural space for solutions to be in is

$$u \in L^\infty(L^2) \qquad \text{and} \qquad \nabla u \in L^2(L^2).$$

In particular, this implies that for any finite $T > 0$, the solution is naturally $L^2_T(H^1)$, although $\|u\|_{L^2_T(H^1)}$ may depend on T. This is a stark contrast with the Euler equations, where the energy estimates had only given an $L^\infty(L^2)$ bound which was insufficient to construct solutions. Here, solutions are naturally $L^2_{\mathrm{loc}}(H^1)$, which will give enough compactness to construct weak solutions. Another difference is that (4.2) allows to bound the order one $L^2_T(H^1)$ norm of u through the order zero norm $\|u_0\|_{L^2}$ on the initial datum. This is a feature of viscosity, which naturally places the solution in a higher regularity space than the initial datum.

With these remarks in mind, we turn our attention to weak solutions. Before examining their existence, we define the notion of weak Leray solution.

Definition 41. *Consider a divergence-free vector field $u_0 \in L^2$. We say that $u \in L^2_{\mathrm{loc}}(L^2)$ is a weak Leray solution of the Navier-Stokes* (4.1) *equations related to the initial datum u_0 if the following conditions are satisfied:*

(i) we have $u \in L^\infty(L^2)$ *and* $\nabla u \in L^2(L^2)$ *;*

(ii) the divergence equation is solved in the sense of distributions $\operatorname{div}(u) = 0$ *in* $\mathcal{D}'(\mathbb{R}_+^\times \mathbb{R}^d)$ *;*

(iii) the first equation of (4.1) *is solved in the following sense: for any vector field* $\phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$ *such that* $\operatorname{div}(\phi) = 0$*, we have*

$$\int_0^{+\infty} \int \Big\{ \partial_t \phi \cdot u + \nabla\phi : u \otimes u + \Delta\phi \cdot u \Big\} \, \mathrm{d}x \, \mathrm{d}t = \int \phi(0,x) u_0(x) \, \mathrm{d}x; \tag{4.3}$$

(iv) u satisfies an energy inequality: for (almost) every $T > 0$*,*

$$\frac{1}{2}\|u(T)\|_{L^2}^2 + \int_0^T \int |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \le \frac{1}{2}\|u_0\|_{L^2}^2.$$

Moreover, we say that u satisfies the strong energy inequality if we also have, for all $t_1 < t_2$*,*

$$\frac{1}{2}\|u(t_2)\|_{L^2}^2 + \int_{t_1}^{t_2} \int |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \le \frac{1}{2}\|u(t_2)\|_{L^2}^2.$$

The notion of strong energy inequality will be useful later on.

The reader should note that the pressure is not involved at all in the definition above. This fact deserves a few remarks. First of all, if $\phi \in \mathcal{D}(\mathbb{R}^d)$ is a vector field with $\operatorname{div}(\phi) = 0$, then integration by parts gives, for any $\pi \in \mathcal{D}'(\mathbb{R}^d)$,

$$\langle \nabla\pi, \phi \rangle = -\langle \pi, \operatorname{div}(\phi) \rangle = 0.$$

This is why π does not appear in the weak formulation (4.3). Conversely, if $f \in \mathcal{S}'$ is a distribution such that

$$\operatorname{div}(\phi) = 0 \quad \Rightarrow \quad \langle f, \phi \rangle = 0$$

for all $\phi \in \mathcal{D}$, then it is possible to find a $g \in \mathcal{S}'$ such that $f = \nabla g$.

Remark 42. While this remark is psychologically reassuring (in that it always seems possible to reconstruct the pressure from the solution) it hides the fact that a bit more work is actually needed to do just that. Formally, the property above shows, at every time $t \ge 0$, the existence of a pressure $\pi(t) \in \mathcal{S}'$ such that

$$\partial_t u(t) + (u\cdot) u(t) + \nabla\pi(t) = \Delta u(t).$$

However, π has no reason *a priori* to define a distribution in time and space $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{R}^d)$, because nothing is known about its time regularity. The reader should keep in mind that functions of the form $1/|t|$ do *not* define distributions.. In practice, this issue is very secondary to the study of the Navier-Stokes equations: a pressure solving (4.1) can always be constructed from a Leray solution by means of Fourier multipliers

$$\pi = (-\Delta)^{-1} \partial_j \partial_k (u_j u_k).$$

We refer to the classical textbook [17], Chapters 5 and 11, where the space $\mathcal{T}'$ of spatially tempered distributions is introduced to address this issue. We will not mention it further and be content to forget about the pressure in what follows.

Another important point concerns the regularity of Leray solutions. The reader may have noticed that points *(i)* and *(iv)* are not necessary to define a notion of weak solution for NS. However, they are included in the definition for the main reason that weak solutions that are not Leray solutions have a very pathological behavior, as shown by the following theorem.

Theorem 43 (Buckmaster, Vicol, [6] 2019)**.** *Let $d = 3$ and consider the NS equations on the periodic setting $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$. There exists $\beta > 0$ such that for any $T > 0$ and any nonnegative continuous function $e : [0, T] \longrightarrow \mathbb{R}_+$ there exists*

(i) an divergence-free initial datum $u_0 \in L^2(\mathbb{T}^3)$,

(ii) a weak solution $u \in C^0_T\big(H^\beta(\mathbb{T}^3)\big)$ of NS associated to u_0,

such that $\|e(t)\|_{L^2} = e(t)$ for all $t \in [0, T]$.

In particular, weak solutions of NS are not in general unique. This theorem means that it is perfectly possible to start with an initial datum $u_0 = 0$ and "create" energy out of nowhere if the solution is not assumed to be Leray. The proof of this result relies on a method called *convex integration*, which has been applied to a wide family of PDEs to show that weak solutions can be too weak to get uniqueness.

On the other hand, the extra assumptions that make a weak solution a Leray solution are not costly: for any divergence-free $u_0 \in L^2$, Leray solutions always exist.

Theorem 44 (Leray, [23], 1934)**.** *Let $d \in \{2, 3\}$ and $u_0 \in L^2$ be a divergence-free initial datum. Then there exists an associated weak Leray solution. Furthermore, u satisfies the strong energy inequality.*

The proof of existence of weak Leray solutions is not difficult. It is very similar to what we have already done: definition of an approximate system through Fourier projection, Cauchy-Lipschitz theorem, compactness, *etc.* We therefore skip it and refer to [17] (the original article of Leray[1] [23] is in French).

Remark 45. The theorem of Leray is stated for dimensions $d = 2$ or $d = 3$, but it also holds when $d = 4$. In the rest of this chapter, we will limit our attention to $d \in \{2, 3\}$.

4.2 Uniqueness of Solutions

In this section, we examine the uniqueness of solutions by means of formal computations. However, all the energy estimates and integration by parts necessary for these to be fully justified is not too difficult, although tedious. It is mainly a matter of using the appropriate test functions in (4.3).

To establish uniqueness of solutions, we seek a stability estimate. Consider therefore $u_{0,1}$ and $u_{0,2}$ two L^2 divergence-free initial date and u_1, u_2 two associated weak Leray solutions. We note $\delta u = u_2 - u_1$ and $\delta u_0 = u_{0,2} - u_{0,1}$. By forming the difference of the equations solved by u_2 and u_1, we see that

$$\begin{cases} \partial_t(\delta u) + (u_2 \cdot \nabla)\delta u + (\delta u \cdot \nabla)u_1 + \nabla(\delta\pi) = \Delta(\delta u) \\ \operatorname{div}(\delta u) = 0. \end{cases}$$

In the above, $\delta\pi = \pi_2 - \pi_1$ is the formal difference of pressures which does not in reality appear in the definition of Leray solutions, and is not involved in the computations. By taking the scalar product of the first equation by δu and integrating, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\delta u|^2 + \int_0^t \int |\nabla \delta u|^2 = -\int (\delta u \cdot \nabla)u_1 \cdot \delta u. \tag{4.4}$$

The goal is to estimate the righthand side integral in order to obtain a differential inequality and apply Grönwall's lemma. We will need the following refined Sobolev embeddings.

[1] It is worth noting that Leray's work in hydrodynamics predates the war. When he was captures by the Nazis, and wishing to avoid working for the development of weapons, he claimed being a topologist, which he actually ended up becoming. The notion of sheaf is due to him.

Proposition 46 (see Theorem.66 in [2])**.** *Consider* $0 \le s < d/2$ *and* $p \in [2, +\infty[$ *such that*

$$\frac{s}{d} + \frac{1}{p} = \frac{1}{2}.$$

Then we have a continuous embedding $H^s \subset L^p$.

Remark 47. In this proposition, it is important to notice that it is impossible to take $p = +\infty$ (or equivalently $s = d/2$). We have already noted that the space $H^{d/2}$ is *not* included in L^∞. Rather, we have the weaker embedding in the space of functions of bounded mean oscillations $H^{d/2} \subset \mathrm{BMO}$.

4.2.1 Uniqueness of Leray Solutions in 2D

In this paragraph, we use these Sobolev embeddings to prove uniqueness of 2D Leray solutions. By using Proposition 46, we see that $H^{1/2}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$ so that Hölder's inequality provides

$$\begin{aligned}\left| \int (\delta u \cdot \nabla) u_1 \cdot \delta u \, \mathrm{d}x \right| &\le \|\nabla u_1\|_{L^2} \|\delta u\|_{L^4}^2 \\ &\le C \|\nabla u_1\|_{L^2} \|\delta u\|_{H^{1/2}}^2.\end{aligned}$$

In order to estimate the $H^{1/2}$ norm, we make use of the interpolation inequality from Proposition 30, which provides $\|f\|_{H^{1/2}} \le \|f\|_{L^2}^{1/2} \|f\|_{H^1}^{1/2}$. We see that

$$\begin{aligned}\left| \int (\delta u \cdot \nabla) u_1 \cdot \delta u \, \mathrm{d}x \right| &\le C \|\nabla u_1\|_{L^2} \|\delta u\|_{L^2} \|\delta u\|_{H^1} \\ &\le C \|\nabla u_1\|_{L^2} \|\delta u\|_{L^2} \Big(\|\delta u\|_{L^2} + \|\nabla \delta u\|_{L^2} \Big).\end{aligned}$$

Here, we introduce a trick that is very often used in such estimates. Remember that our goal is to find a differential inequality in order to apply Grönwall's lemma. Since the lefthand side of (4.4) contains a derivative of $\|\delta u\|_{L^2}^2$, it follows that we should be looking for a righthand side bound involving only $\|\delta u\|_{L^2}^2$, whereas inequality immediately above also involves $\|\nabla \delta u\|_{L^2}$. The way we solve this problem is by taking advantage of the fact that the lefthand side of (4.4) contains a $\|\delta u\|_{L^2}^2$ term (from viscosity effects), and we will use it to "absorb" the gradient terms. More precisely, note that because $(x-y)^2 = x^2 + y^2 - 2xy \ge 0$, we have the inequality

$$ab = \sqrt{\epsilon} a + \frac{1}{\sqrt{\epsilon}} b \le \frac{1}{2} \epsilon a^2 + \frac{1}{2\epsilon} b^2.$$

Therefore, we may write, for any $\epsilon > 0$,

$$\left| \int (\delta u \cdot \nabla) u_1 \cdot \delta u \, \mathrm{d}x \right| \le C \|\nabla u_1\|_{L^2} \|\delta u\|_{L^2}^2 + C\epsilon \|\nabla \delta u\|_{L^2}^2 + \frac{C}{\epsilon} \|\nabla u_1\|_{L^2}^2 \|\delta u\|_{L^2}^2.$$

We choose for ϵ the value $\epsilon = \frac{1}{2C}$. By plugging the resulting inequality in (4.4), and subtracting ("absorbing") both terms by $\frac{1}{2} \|\nabla \delta u\|_{L^2}^2$, we finally obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\delta u\|_{L^2}^2 \le C \Big(\|\nabla u_1\|_{L^2} + \|\nabla u_1\|_{L^2}^2 \Big) \|\delta u\|_{L^2}^2,$$

so that a direct application of Grönwall's inequality gives

$$\|\delta u(T)\|_{L^2}^2 \le \|\delta u_0\|_{L^2}^2 \exp \Big(C \|\nabla u_1\|_{L^1_T(L^2)} + \|\nabla u_1\|_{L^2_T(L^2)}^2 \Big),$$

and a final application of the energy inequality for u_1 (see point *(iv)* in Definition 41) gives the stability estimate

$$\|\delta u(T)\|_{L^2}^2 \leq \|\delta u_0\|_{L^2}^2 \exp\left(C\sqrt{T}\|u_{0,1}\|_{L^2} + \|u_{0,1}\|_{L^2}\right),$$

and therefore uniqueness. These are the main arguments that enter the proof of the following theorem.

Theorem 48. *Consider the 2D Navier-Stokes equations $d = 2$. For any divergence-free $u_0 \in L^2(\mathbb{R}^2)$, there exists a unique weak Leray solution associated to u_0.*

The proof of this result is essentially contained in the computations above. Of course, everything we have written has been quite formal, so that a rigorous proof would have to make use of the weak formulation (4.3) by choosing $\phi = \delta u$ and making sure all the brackets are well-defined and all the integration by parts are justified.

4.2.2 The Case of Three Dimensions

Uniqueness of 3D Leray solutions is still unknown when $d = 3$, although the past years have seen spectacular results.

Theorem 49 (Albritton, Brué, Colombo, [1], 2022)**.** *There exists a $T > 0$ and a finite energy force $f \in L^1_T(L^2)$ such that the problem*

$$\begin{cases} (\partial_t - \Delta)u + (u \cdot \nabla)u + \nabla\pi = f \\ \operatorname{div}(u) = 0 \end{cases}$$

has two distinct Leray solutions with the initial datum $u_0 = 0$.

Though this does not bode well for the uniqueness of Leray solutions, we should point out that the force term f is highly irregular, and that it it a crucial element of the proof. This means that this theorem, although very impressive, still does not give a method to show non-uniqueness of weak Leray solutions.

Let us come back to the question of uniqueness of Leray solutions. Trying to use the same method as in 2D to prove uniqueness cannot work, because the Sobolev embedding $H^{1/2}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$ fails in 3D. Instead, we prove a conditional uniqueness result: we will see that a 3D Leray solution that has some additional regularity is unique.

Theorem 50. *Consider $u_0 \in L^2(\mathbb{R}^3)$ a divergence-free function and u an associated Leray solution. Moreover, assume that $u \in L^4_{\text{loc}}(H^1)$. Then u is the only Leray solution with initial datum u_0.*

Remark 51. Note that the solution in the theorem is more regular with respect to the time variable, but there is no added regularity with respect to space.

The (formal) proof of this result relies on computations similar to what we did in the 2D case: we begin by writing equation (4.4) and estimate the integral in the righthand side. By using Hölder's inequality, we obtain

$$\left|\int (\delta u \cdot \nabla)u_1 \cdot \delta u \, \mathrm{d}x\right| \leq \|\nabla u\|_{L^2}\|\delta u\|_{L^4}^2.$$

Now, by assumption, the quantity $\|\nabla u_1\|_{L^2}$ is L^4_{loc} with respect to time. This meas that, if we wish to close the stability estimates, we should try to bound $\|\delta u\|_{L^2}^2$ in $L^{4/3}_{\text{loc}}$. We begin by using Proposition 46 to write the Sobolev embedding $H^{3/4} \subset L^4$, as

$$\frac{1}{2} = \frac{1}{4} + \frac{3/5}{3}.$$

We follow up with an interpolation inequality (see Proposition 30). By using the convex combination $\frac{3}{4} = 0 \times \frac{1}{4} + 1 \times \frac{3}{4}$, we see that

$$\|\delta u\|_{L^4} \leq C\|\delta u\|_{H^{3/4}} \leq C\|\delta u\|_{L^2}^{1/4}\|\delta u\|_{H^1}^{3/4}.$$

By plugging this in (4.4), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\delta u|^2 \,\mathrm{d}x + \int |\nabla \delta u|^2 \,\mathrm{d}x \leq C\|\nabla u_1\|_{L^2}\|\delta u\|_{L^2}^{1/2}\|\delta u\|_{H^1}^{3/2}.$$

Just as in the 2D case, we will seek to absorb a part of the product in the upper bound in the second integral in the lefthand side. Here, we will Young's inequality, which implies that for $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p, q < +\infty$, we have $xy \leq x^p + y^q$, and so

$$\forall a, b \in \mathbb{R}_+, \qquad ab = \epsilon^{1/p} a \frac{1}{\epsilon^{1/p}} b \leq \epsilon a^p + \frac{1}{\epsilon^{q/p}} b^q.$$

Young's inequality can easily be proved by means of a convexity inequality. By using $q = 4$ and $p = 3/4$, we deduce

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\delta u|^2 \,\mathrm{d}x + \int |\nabla \delta u|^2 \,\mathrm{d}x &\leq \epsilon\|\delta u\|_{H^1}^1 + C(\epsilon)\|\nabla u_1\|_{L^2}^4\|\delta u\|_{L^2}^2 \\ &\leq \left(C(\epsilon)\|\nabla u_1\|_{L^2} + \epsilon\right)\|\delta u\|_{L^2}^2 + \epsilon\|\nabla \delta u\|_{L2}^2.\end{aligned}$$

If we choose $\epsilon = \frac{1}{2}$, then we may subtract both terms in the inequality above by $\frac{1}{2}\|\nabla \delta u\|_{L^2}^2$, and then use Grönwall's inequality to finally get a stability estimate:

$$\|\delta u\|_{L^2}^2 \leq \|\delta u_0\|_{L^2}^2 \exp\left(C(\epsilon)t + \int_0^t \|\nabla u_1\|_{L^2}^4\right). \tag{4.5}$$

This shows that Leray solutions are indeed unique when they have additional $L^4_{\mathrm{loc}}(H^1)$ regularity.

Remark 52. Note how the final estimate above only depends on $\|\nabla u_1\|_{L^2}$. Theorem 50 is called a *weak-strong* uniqueness result. In other words, if a Leray solution u_1 is "strong" $L^4_{\mathrm{loc}}(H^1)$, then any other "weak" Leray solution with the same initial datum must be equal to u_1. In general, weak-strong uniqueness results are a bit tricky to prove rigorously, because they are concerned with irregular solutions. Here, our computations were essentially formal, and would have to be fully justified to recover the full statement of Theorem 50.

4.3 Existence of a Strong Unique Solution in 3D

Now that we have found a condition under which solutions are unique, we would like to show that it is non-empty: that there indeed are solutions that have that kind of regularity. This is the purpose of the Fujita-Kato theorem.

Theorem 53 (Fujita, T. Kato, [16], 1964)**.** *Consider $u_0 \in H^{1/2}(\mathbb{R}^3)$ a divergence free vector field. There exists a $T > 0$ and a weak Leray solution u such that $u \in L^4_T(H^1)$. Moreover, there exists an absolute constant $\eta > 0$ such that if $\|u_0\|_{H^{1/2}} \leq \eta$, then T can be chosen infinite $T = +\infty$.*

Notice how this result is very different from what we have obtained so far from Euler or SQG. This is due to the fact that the Navier-Stokes are essentially parabolic, as we will see in the next paragraph.

4.3.1 Using a Fixed-Point Theorem

The main idea to prove the Fujita-Kato theorem is the use of a fixed point method. While in the two previous chapters we had essentially used orthogonal projection operators to construct approximate solutions, this is no longer possible: the space $L^4(H^1)$ is not Hilbert, so there is no notion of orthogonal projection.[2] This is why a fixed point method is more appropriate. To define the fixed point operator, we use the heat equation structure: write the NS equations as

$$(\partial_t - \Delta)u = -\mathbb{P}\big((u\cdot\nabla)u\big),$$

so that Duhamel's formula gives an expression of the solution as a function of itself:

$$u(t) = \underbrace{e^{t\Delta}u_0}_{:=a} \underbrace{- \int_0^t e^{(t-t')\Delta}\mathbb{P}\big((u\cdot\nabla)u\big)(t')\mathrm{d}t'}_{:=B(u,u)}.$$

In the above, $e^{t\Delta}$ refers to the heat operator semi-group: the function $e^{t\Delta}f$ is the solution of the homogeneous heat equation with initial datum f, and has Fourier transform $e^{-t|\xi|^2}\widehat{f}(\xi)$. In order to apply the fixed point theorem, we will be looking for a Banach space X and a closed ball $\overline{B_X}(0,R)$ such that the map

$$\Psi : \begin{array}{c} \overline{B_X}(0,R) \longrightarrow \overline{B_X}(0,R) \\ : u \longmapsto a + B(u,u) \end{array}$$

is a contraction. In particular, we will need three key ingredients:

1. Show that the bilinear map $B : X\times X \longrightarrow X$ is continuous ;
2. Show that $\overline{B_X}(0,R)$ is stable under Ψ ;
3. Show that Ψ is indeed a contraction.

We will prove the first of these by exploring basic properties of the heat equation. We therefore assume for now that B is bounded as a bilinear operator, so that it has a finite norm $\|B\|$. Now, for $u_i n\overline{B_X}(0,R)$, the inequalities

$$\begin{aligned} \|\Psi(u)\|_X &\leq \|a\|_X + \|B\|\|u\|_X^2 \\ &\leq \|a\|_X + R^2\|B\| \end{aligned}$$

show that Ψ stabilizes the ball under the sufficient condition that $\|a\|_X \leq \frac{1}{2}R$ and $R \leq \frac{1}{2\|B\|}$. Such a radius will certainly exists as soon as the condition

$$\|a\|_X < \frac{1}{4\|B\|} := \eta \tag{4.6}$$

holds. Finally, it is easy to see that Ψ is a contraction under the above condition: let $u, v \in \overline{B_X}(0,R)$, so that

$$\begin{aligned} \|\Psi(u)-\Psi(v)\|_X &\leq \|B\|\big(\|u\|_X + \|v\|_X\big)\|u-v\|_X \\ &\leq 2R\|B\|\|u-v\|_X. \end{aligned}$$

Thanks to (4.6) the radius R can be chosen so that $2R\|B\| < 1$, and the map Ψ is indeed a contraction.

[2] In general, projections in non-Hilbert Banach spaces may suffer from terrible pathologies. For example, it can be proved that any Banach space that is not (quasi-isometrically) isomorphic to a Hilbert space contains a closed subspace on which there is no continuous projection.

In conclusion, we see that if $\|e^{t\Delta}u_0\|_X$ is small enough, then the fixed point theorem applies and provides a solution $u \in X$. Keeping in mind that we should choose $X = L^4_T(H^1)$ (or a related space) to prove the Fujita-Kato theorem, there are two ways to realize this condition. The first way is to take u_0 small enough in an appropriate space so that the condition $\|e^{t\Delta}u_0\| \leq \eta$ will hold independently of T, and this provides global solutions. And the other way is to notice that we expect to have $\|e^{t\Delta}u_0\|_{L^4_T(H^1)} \longrightarrow 0$ as $T \to 0^+$, so that choosing T small enough will also realize the condition.

4.3.2 Introduction to Homogeneous Spaces

Before moving on to the bulk of the proof of the Fujita-Kato theorem, we will introduce a new tool: homogeneous Sobolev spaces. These spaces address some of the limitations of the usual Sobolev spaces. For example, whenever one wishes to perform a H^k estimate on a function, there are two bounds to compute:

$$\|f\|_{H^k} \approx \|f\|_{L^2} + \|\nabla^k f\|_{L^2}.$$

The first quantity is the L^2 norm of f and the second quantity is the L^2 norm of $\nabla^k f$. However, this double estimation introduces needless technical difficulties in the proofs. Notice for example how in (4.5) only the norm $\|\nabla u_1\|_{L^2}$ appeared in the estimate, and how we in fact had no need to assume that $u_1 \in L^4(H^1)$, but that the simpler condition $\nabla u_1 \in L^4(L^2)$ was enough.

However, homogeneous spaces are not simply born out of a desire so have simpler computations. They also possess a property which the spaces H^k lack: *scaling*.[3]

Definition 54. *Let $X \subset \mathcal{S}'$ be a Banach space of distributions. We say that X scales as $\alpha \in \mathbb{R}$ if the relation*

$$\big\|f(\lambda x)\big\|_X = \lambda^\alpha \|f\|_X$$

holds for every $f \in X$ and $\lambda > 0$.

Example 55. The space $L^2(\mathbb{R}^d)$ scales as $-d/2$. A change of variables provides $\big\|f(\lambda x)\big\|_{L^2} = \lambda^{-d/2}\|f\|_{L^2}$.

Example 56. For the same reasons, the norm $\|\nabla^k f\|_{L^2}$ scales as $k - d/2$, although it is not (yet) associated to a Banach space.

Example 57. On the other hand, the Sobolev space H^k has no scaling properties if $k \geq 1$ is an integer, as

$$\big\|f(\lambda x)\big\|_{H^k} \approx \lambda^{-d/2}\|f\|_{L^2} + \lambda^{k-d/2}\|\nabla^k f\|_{L^2}.$$

However, the relation above gives an interesting insight on the meaning of scaling properties. Note how the L^2 norm $\lambda^{d/2}\|f\|_{L^2}$ dominates the expression when $\lambda \ll 1$, whereas $\lambda^{k-d/2}\|\nabla^k f\|_{L^2}$ dominates when $\lambda \gg 1$. This means that when "zooming out" ($\lambda \ll 1$), $\|f\|_{L^2}$ dictates the large scale behavior of the function such as decay at infinity, and when "zooming in" ($\lambda \gg 1$) $\|\nabla^k f\|_{L^2}$ dictates the small scale behavior of the function (*i.e.* regularity). In a function space that has a scaling property, the large scale and small scale behavior are closely linked.

There are two main reasons why this notion of scaling is extremely important in analysis (and in particular in the analysis of PDEs).

1. The first reason is that the scaling of a function space is intimately related to its embedding properties. Assume that X and Y are two function spaces such that the continuous

[3] I am deeply indebted to Lorenzo Pompili for his insight and long explanations on scaling while I was preparing this lecture.

embedding $X \subset Y$ holds, and further assume that X and Y scale as α and β respectively. Then, for any $f \in X$;

$$\big\|f(\lambda x)\big\|_Y = \lambda^\beta \|f\|_Y \leq C\lambda^\beta \|f\|_X = C\lambda^{\beta-\alpha}\big\|f(\lambda x)\big\|_X.$$

Consequently, for the embedding to hold, we must have $\alpha = \beta$. Two function embedded spaces that have a scaling property must share the same scaling! For example, this means that $L^p \not\subset L^q$ whenever $p \neq q$.

2. The second reason for which scaling is so important is that the Navier-Stokes equations (and many other PDEs) have a natural scaling. For any function $f(t,x)$, we denote $f_\lambda(t,x) := \lambda f(\lambda^\alpha t \lambda^\beta x)$. Now, if a function $u(t,x)$ is a solution of the NS equations, the function u_λ also is a solution at the condition that

$$\partial_t u_\lambda + \mathbb{P}\big((u_\lambda \cdot \nabla)u_\lambda\big) - \Delta u_\lambda = \lambda^\alpha (\partial u)_\lambda + \lambda^{1+\beta}\left(\mathbb{P}\big((u\cdot\nabla)u\big)\right)_\lambda - \lambda^{2\beta}(\Delta u)_\lambda = 0,$$

(the operator $\mathbb{P}$ does not change the scaling, as its symbol is a homogeneous function of degree zero). In particular, if $\alpha = 2$ and $\beta = 1$, then $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$ is a solution of the NS equations. In general, it is better to work function spaces X whose scaling is compatible with the Navier-Stokes equations, so that $\|u_\lambda\|_X = \|u\|_X$.

For these reasons, we will try to replace the Sobolev spaces H^k by spaces that have scaling properties.

Definition 58. *Consider $s < d/2$. The space $\dot{H}^s(\mathbb{R}^d)$ is the set of distributions $f \in \mathcal{S}'$ such that $\widehat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and*

$$\|f\|_{\dot{H}^s} := \left(\int |\xi|^{2s}\big|\widehat{f}(\xi)\big|\mathrm{d}\xi\right)^{1/2} < +\infty.$$

The space $\dot{H}^s(\mathbb{R}^d)$ is called the homogeneous Sobolev space *of order s. By contrast, the space H^s is sometimes referred to as the* non-homogeneous Sobolev space *of order s.*

We will list a few of the properties of the homogeneous Sobolev space. First of all, these spaces are complete, and even have a hilbertian structure. We encourage the reader to prove this on himself, as it is not too difficult.

Proposition 59 (Proposition 1.34 in [2])**.** *Let $s < d/2$. The space $\dot{H}^s$ is a Hilbert space.*

The space $\dot{H}^s$ also has nice scaling properties: a quick glance at the Fourier transform shows that $\big\|f(\lambda x)\big\|_{\dot{H}^s} = \lambda^{s-d/2}\|f\|_{\dot{H}^s}$. This means that the homogeneous Sobolev space $\dot{H}^s$ and the Lebesgue space L^p share the same scaling when

$$\frac{s}{d} + \frac{1}{p} = \frac{1}{2}. \tag{4.7}$$

The reader should remember that this is precisely the relation used to insure the sharp Sobolev embeddings of Proposition 46. In fact, these embeddings still hold for the homogeneous case.

Proposition 60 (Proposition 1.38 in [2])**.** *Let $s < d/2$ and $p \in [2,+\infty[$ such that (4.7) holds. Then we have the continuous embedding $\dot{H}^s \subset L^p$.*

Interpolation inequalities also hold for homogeneous Sobolev spaces. The proof is very much the same as that of the interpolation inequalities for non-homogeneous spaces.

Proposition 61. *Consider $s_1 < s_2 < d/2$ and a function $f \in \dot{H}^{s_1} \cap \dot{H}^{s_2}$. Then for any $\theta \in [0,1]$ and $s = \theta s_1 + (1-\theta)s_2$, we have $f \in \dot{H}^s$ and the inequality*

$$\|f\|_{\dot{H}^s} \leq \|f\|^\theta_{\dot{H}^{s_1}}\|f\|^{1-\theta}_{\dot{H}^{s_2}}$$

holds.

Finally, to conclude this paragraph, we must warn the reader that homogeneous Sobolev spaces can be extremely tricky to deal with, and sometimes have counter-intuituve properties (as we have already hinted in the previous chapters).

To start with, the condition that $\widehat{f} \in L^1_{\text{loc}}(\mathbb{R}^d)$ is essential. Consider for example the space $\dot{H}^1(\mathbb{R}^3)$. Then the Fourier-Plancherel identity shows that $\|f\|_{\dot{H}^1} = \|\nabla f\|_{L^2}$. Consider the function $f = 1$, whose Fourier transform $\widehat{f} = \delta_0$ the Dirac delta, then formally $\|1\|_{\dot{H}^1} = \|\nabla 1\|_{L^2} = 0$. The local integrability condition excludes functions that might cancel the homogeneous norm.

Next, we should point out that the condition $s < d/2$ is all important. In fact, the space of $f \in \mathcal{S}'$ such that $\widehat{f} \in L^1_{\text{loc}}$ and $\|f\|_{\dot{H}^s} < +\infty$ is not complete when $s \geq d/2$ (see Proposition 1.34 in [2]). The reader may wonder whether it is possible to complete that space and still recover a space of distributions, but the answer is that it essentially is not: the completion (which always exists) is naturally a subspace of the quotient of tempered distributions modulo polynomials $\mathcal{S}'/\,\mathbb{R}[X]$.

The topic of homogeneous Sobolev spaces when $s \geq d/2$ is rather technical, and has few applications in PDEs. The interested reader can find more in [4], [11]. However, the main takeaway is that homogeneous spaces should be dealt with very cautiously.

4.3.3 Proof of the Fujita-Kato Theorem

What is left to do is to show that the bilinear operator $B : X \times X \longrightarrow X$ is bounded for some choice of X. Because we already have a uniqueness result for Leray solutions with $\nabla u \in L^4_T(L^2)$, we will choose $X = L^4_T(\dot{H}^1)$, which is a space that is invariant under the scaling of the Navier-Stokes equations. Most of our estimates will rely on the following heat kernel lemma.

Lemma 62. *Consider a biven force f and v a solution of the heat equation*

$$\begin{cases} (\partial_t - \Delta)v = f \\ v(0) = v_0. \end{cases}$$

Then, for all $p \in [2, +\infty]$ and $s \in \mathbb{R}$ such that $s + 2/p < d/2$, we have the inequality

$$\|v\|_{L^p_T(\dot{H}^{s+d/p})} \leq \|v_0\|_{\dot{H}^s} + C\|f\|_{L^2_T(\dot{H}^{s-1})}.$$

Proof. Take the Fourier transform of the heat equation:

$$\partial\widehat{v}(t,\xi) + |\xi|^2\widehat{v}(t,\xi) = \widehat{f}(t,\xi).$$

By multiplying by $|\xi|^{2s}$ and integrating with respect to the Fourier variable, using the Cauchy-Schwarz inequality and Young's inequality, we obtain, for every $\epsilon > 0$,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\xi|^{2s}\big|\widehat{v}(t,\xi)\big|^2\mathrm{d}\xi + \int |\xi|^{2s+2}\big|\widehat{v}(t,\xi)\big|^2\mathrm{d}\xi &= \int |\xi|^{2s}\widehat{f}(t,\xi)\cdot\widehat{v}(t,\xi)\mathrm{d}\xi \\ &\leq \left(\int |\xi|^{2s-2}\big|\widehat{f}(t,_x i)\big|^2\mathrm{d}\xi\right)^{1/2}\left(\int |\xi|^{2s+2}\big|\widehat{v}(t,\xi)\big|^2\mathrm{d}\xi\right)^{1/2} \\ &\leq \epsilon\|v\|^2_{\dot{H}^{s+1}} + \frac{1}{\epsilon}\|f\|^2_{\dot{H}^{s-1}}. \end{aligned}$$

By absorbing the factor of ϵ in the lefthand side, we find a differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{\dot{H}^s} + \|v\|^2_{\dot{H}^{s+1}} \leq C\|f\|^2_{\dot{H}^{s-1}}.$$

Integrating this relation, or instead using Grönwall's inequality gives the conclusion of the lemma in the case where $p = 2$ or $p = +\infty$,

$$\begin{aligned} &\|v\|_{L^2_T(\dot{H}^{s+1})} \leq C\|f\|_{L^2_T(\dot{H}^{s-1})} \leq \|v_0\|_{\dot{H}^s} + \|f\|_{L^2_T(\dot{H}^{s-1})} \\ &\|v\|_{L^\infty_T(\dot{H}^s)} \leq \|v_0\|_{\dot{H}^s} + \|f\|_{L^2_T(\dot{H}^{s-1})}. \end{aligned}$$

In order to conclude, we use an interpolation inequality. Thanks to the convex combination $s + \frac{2}{p} = (1 - \frac{2}{p})s + \frac{2}{p}(s+1)$, we write...

`[WORK IN PROGRESS]`

$\square$

Appendix A

Survival Kit on Weak Convergence

A.1 Introduction: Strong Compactness

This short section covers a few notions on weak convergence. The goal is not to provide a thorough presentation of the topic, but merely to give intuition on this notion through a few statements and examples. Most of the proofs will be ignored, and we refer instead to the standard textbook [5], which covers more than what is needed to follow these notes.

As the student will probably have understood from the proofs in these notes, compactness is one of the main tools used to construct solutions of PDEs, and finding compactness properties of a family (f_n) (say of approximate solutions) is the principle challenge of a proof. Said compactness is usually obtained through uniform bounds: assume there is a Banach space X such that

$$\|f_n\|_X \leq R$$

for some constant $R > 0$. Then, if the ball $B(0,R) \subset X$ can be compactly embedded in a Banach (or metric) function space Y, one may extract a limit in Y. There is an extraction $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ and a $f \in Y$ such that

$$f_{\phi(n)} \longrightarrow f \qquad \text{in } Y.$$

For example, take $X = H^1$. Then, the Rellich-Kondrachov theorem states that the embedding $H^1 \subset L^2_{\text{loc}}$ is compact.

Theorem 63 (Rellich-Kondrachov (Theorem 3.16 in [5]))**.** *The embedding $H^1 \subset L^2_{\text{loc}}$ is compact. In other words, any bounded subset $A \subset H^1$ is a relatively compact subset of L^2_{loc}.*

In the above, the space L^2_{loc} is the space of locally L^2 functions on $\mathbb{R}^d$. A measurable function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is locally L^2 if the function $\mathbb{1}_K f$ is L^2 for any compact subset $K \subset \mathbb{R}^d$. Note that the space L^2 is *not* a Banach space. Instead, its topology is defined by a metric:

$$d(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \min\left\{1, \|f-g\|_{L^2(B_k)}\right\},$$

where $B_k = B(0,k)$ are balls of increasing radius $k \geq 1$ whose union cover the whole space $\mathbb{R}^d = \bigcup_k B_k$. A sequence of functions (f_n) converges in L^2_{loc} to $f \in L^2_{\text{loc}}$ if and only if the convergence

$$\mathbb{1}_K f_n \longrightarrow \mathbb{1}_K f \qquad \text{in } L^2$$

holds for all compact subset $K \subset \mathbb{R}^d$. Examples of L^2_{loc} functions include (say of $d = 1$) e^x, polynomials, $x^{\epsilon-1/2}$, *etc.*

Example 64. Consider the sequence of functions $f_n(x) = \exp(-(x-n)^2)$ defined on the real line. It is bounded in H^1_{loc}, and converges locally to zero in L^2_{loc}: for any compact interval $I \subset \mathbb{R}$,

$$\exp(-(x-n)^2) \longrightarrow 0 \qquad \text{in } L^2(I).$$

However, note that (f_n) does not converge to zero in $L^2(\mathbb{R})$, because it is of constant norm $\|f_n\|_{L^2} = C > 0$.

A.2 Presenting Weak Convergence

We continue to explore the question of extracting a converging sequence from a bounded sequence (f_n) in a Banach space X. Assume again that X is compactly embedded in a (metric) vector space Y, so that we have convergence of an extraction

$$f_{\phi(n)} \longrightarrow f \qquad \text{in } Y.$$

Several questions emerge. Firstly, what can be said about the limit f? Obviously, it is an element of Y, but since Y is a larger space than X, we could hope for more. Especially as the sequence (f_n) remains bounded in X, it seems intuitive that the limit f somehow should also be an element of X. This is generally true.

Let us see how this works on an example. We look again at the situation from the previous section: let (f_n) be a bounded sequence of functions in H^1. We know that an extraction converges in L^2_{loc},

$$f_{\phi(n)} \longrightarrow f \qquad \text{in } L^2_{\mathrm{loc}}. \tag{A.1}$$

In particular, for any smooth and compactly supported $\psi \in \mathcal{D}$, we have

$$\int f_{\phi(n)}(x)\psi(x)\,\mathrm{d}x \longrightarrow \int f(x)\psi(x)\,\mathrm{d}x.$$

This implies that the limit f satisfies an inequality of the form

$$\sup_{\|\psi\|_{H^{-1}} \le 1} \left| \int f(x)\psi(x)\,\mathrm{d}x \right| \le \varliminf_n \|f_n\|_{H^1} < +\infty,$$

where in the above H^{-1} is the dual space of H^1 (see the remark immediately below). Since $\mathcal{D}$ is a dense subspace of H^{-1}, this means that f must be an element of H^1.

Remark 65. The Sobolev space H^1 is a Hilbert space, so the reader may be surprised to see that the dual space $H^{-1} = (H^1)'$ is considered to be a different space than H^1. This is a subtle but important point. The space H^1 is a Hilbert space for the scalar product

$$\langle f, g \rangle_{H^1} := \int \big(fg + \nabla f \cdot \nabla g\big)\,\mathrm{d}x.$$

The Riesz representation theorem implies that any bounded linear map $T \in (H^1)'$ can be represented by a function $g \in H^1$ through the formula $T(f) = \langle g, f \rangle_{H^1}$. However, the bounded linear map T also defines a distribution $h \in \mathcal{D}'$ through the relation

$$\forall \psi \in \mathcal{D}, \qquad T(\psi) := \langle h, \psi \rangle_{\mathcal{D}' \times \mathcal{D}}. \tag{A.2}$$

If h were a locally integrable function, the bracket above would be equal to $T(\psi) = \int h\psi$. The issue is that h and g are not equal as distributions, since they are not defined by the same formula. In fact, while g is a H^1 function, the distribution T is in general the derivative of a L^2 function. The set of distributions T such that the map (A.2) is bounded on H^1 is called H^{-1}. Both H^{-1} and H^1 are isometric to the dual space H^1 (and so to each other), but they are not identical as spaces of distributions!

We continue with this example. We have proven that the L^2_{loc} limit f still is a H^1 function. This is more than the simple L^2_{loc} convergence (A.1) gives by itself, and follows from the fact that (f_n) is a bounded sequence of H^1. The next natural question is then whether the sequence converges to f in some stronger topology than L^2_{loc}. Again, the answer is yes.

Definition 66. *Consider H a Hilbert space and (f_n) a sequence of elements of H. We say that (f_n) converges weakly in H to an element $f \in H$ if*

$$\forall g \in H, \qquad \langle f_n, g\rangle_H \longrightarrow \langle f, g\rangle_H.$$

This is noted $f_n \rightharpoonup g$ (in H).

This convergence is called *weak*, whereas convergence in the norm topology of H is *strong*. To provide intuition, let us give a couple of examples.

Example 67. Consider again the sequence $f_n(x) = \exp(-(x-n)^2)$. Then $f_n \rightharpoonup 0$ in H^1. However, the sequence (f_n) does not converge strongly in H^1.

Example 68. Consider the sequence $f_n(x) = \exp(inx) \in L^2(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Then it follows from the Riemann-Lebesgue theorem that $f_n \rightharpoonup 0$ in $L^2(\mathbb{T})$. Again, the sequence (f_n) does not converge strongly in $L^2(\mathbb{T})$.

The weak topology of H is the coarsest topology of H such that all linear maps $T \in H'$ are continuous. The main interest of the weak topology of a Hilbert space H is that is turns H into a locally compact topological space.

Theorem 69. *Let H be a (not necessarily separable) Hilbert space and (f_n) a bounded sequence of elements of H. Then there exists a $f \in H$ and an extraction $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ such that*

$$f_{\phi(n)} \rightharpoonup f \qquad \text{in } H.$$

Moreover, the norm of the weak limit is bounded by

$$\|f\|_H \leq \overline{\lim_n} \|f_n\|_H.$$

Remark 70. The main issue with the notion of weak convergence is that it is poorly suited to non-linear problems: the function product $(f, g) \mapsto fg$ is usually not continuous for the weak topology. For example, we have seen that $e^{\pm inx} \rightharpoonup 0$ in $L^2(\mathbb{T})$. However, $e^{inx}e^{-inx} = 1$ does not tend to zero.

A.3 Weak and Weak-($*$) Convergence

In the analysis of PDEs, it is very frequent to have to deal with Banach spaces that are not Hilbert. In that case, it is convenient to also define a notion of weak convergence.

Definition 71. *Consider a sequence (f_n) of elements of a Banach space X and let $f \in X$. We say that f_n converges weakly to f in X and note*

$$f_n \rightharpoonup f \qquad \text{in } X$$

if for any bounded linear map $g \in X'$, we have

$$\langle g, f_n\rangle_{X' \times X} \longrightarrow \langle g, f\rangle_{X' \times X}.$$

The weak topology of X, sometimes noted $\sigma(X, X')$, is the coarsest topology on X such that all bounded linear maps are continuous. If $X = H$ is a Hilbert space, then weak convergence as defined immediately above is identical to the notion of weak convergence from the previous paragraph.

The weak topology in Banach spaces possesses very nice properties. For instance, a lower semi-continuous convex function on X (for the norm topology) is also lower semi-continuous for the weak topology. This fact plays an important role in the optimization theory of convex functionals. However, it suffers from a fatal flaw: the weak topology is not, for general Banach spaces, locally compact. The typical example of this bad behavior is $L^1(\mathbb{R}^d)$.

Example 72. Consider the sequence of functions $f_n(x) = \mathbb{1}_{[0,1]}(x-n)$, which is bounded in $L^1(\mathbb{R})$. Let $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ be an extraction. Then $(f_{\phi(n)})$ does not converge for the weak topology. To see this, let $g \in L^\infty(\mathbb{R})$ (recall that $(L^1)' = L^\infty$) be defined by

$$g(x) = \begin{cases} 1 & \text{if } x \in [\phi(k), \phi(k)+1[\text{ when } k \text{ is even} \\ -1 & \text{if } x \in [\phi(k), \phi(k)+1[\text{ when } k \text{ is odd.} \end{cases}$$

Then the brackets

$$\langle g, f_{\phi(n)} \rangle_{L^\infty \times L^1} = \int g(x) f_{\phi(n)}(x)\, \mathrm{d}x = (-1)^n$$

never converge. Consequently, no extraction of the sequence (f_n) converges weakly in L^1. Note that the sequence converges to zero in the weak topology of L^2. In other words, the sequence (f_n) has a natural limit $f = 0$, which is an element of L^1, but the weak topology of L^1 is too strong for the sequence to converge to that limit.

Example 73. Let $\chi \geq 0$ be a smooth function on $\mathbb{R}$ that is supported in $[-1, 1]$ such that $\int \chi = 1$. We define the sequence (f_n) by

$$f_n(x) = n\chi(nx).$$

Then, for any continuous function $\phi \in C^0(\mathbb{R})$, we have the convergence

$$\int f_n(x)\phi(x)\, \mathrm{d}x \longrightarrow \phi(0). \tag{A.3}$$

Here the situation is different. The natural limit of the sequence (f_n) is the Dirac delta δ_0, which is *not* an element of L^1. In a sense, L^1 is not "complete" for the weak convergence. This is another example of bounded sequence which does not have a weak accumulation point in L^1 for the weak convergence.

These two examples show the need for another type of convergence. This will be supplied by the notion of weak-$(*)$ convergence.

Definition 74. *Let X be a Banach space that is the (topological) dual a Banach space Y, so that $X = Y'$ (we say that Y is the* predual *of X). We say that a sequence of elements (f_n) of X converges weakly-$(*)$ to $f \in X$ and note*

$$f_n \overset{*}{\rightharpoonup} f \qquad \text{in } X$$

if for any $g \in Y$ we have

$$\langle f_n, g \rangle_{X \times Y} \longrightarrow \langle f, g \rangle_{X \times Y}.$$

The weak-$(*)$ topology is sometimes noted $\sigma(X, Y)$, and is the topology of bounded linear maps (elements of $Y' = X$) associated to pointwise convergence. Once again, if $X = H$ is a Hilbert space, there is no difference with the notion of weak convergence from the previous paragraph.

The main attribute of the weak-$(*)$ topology is that it turns X into a locally compact space (this is the Banach-Alaoglu-Bourbaki theorem, see Theorem 3.16 in [5]). Moreover, if the predual space Y is separable, then X is sequentially compact for the weak-$(*)$ convergence.

Theorem 75 (Banach-Alaoglu, Corollary 3.30 in [5]). *Consider X a Banach space which has a separable predual Y. Then any bounded sequence (f_n) of elements of X has an accumulation point for the weak-$(*)$ convergence: there is an extraction $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ and a $f \in X$ such that*

$$f_{\phi(n)} \overset{*}{\rightharpoonup} f \qquad \text{in } X.$$

Moreover, the limit satisfies

$$\|f\|_X \leq \overline{\lim_n} \|f_n\|_X.$$

Example 76. The space $X = L^\infty(\mathbb{R})$ has predual $L^1(\mathbb{R})$, which is a separable Banach space. In particular, any bounded sequence of $L^\infty(\mathbb{R})$ functions has a weak-$(*)$ accumulation point. For example, the sequence (f_n) from Example 72 converges weakly-$(*)$ to $f = 0$.

Example 77. Let us look again at the sequence from Example 73, which is bounded in $L^1(\mathbb{R})$. The issue is that the space $L^1(\mathbb{R})$ is not the dual of any Banach space (this is a consequence of the Krein-Millman and Banach-Alaoglu-Bourbaki theorems), so that the weak-$(*)$ topology cannot be defined on $L^1(\mathbb{R})$. A way to overcome this issue is to embed $L^1(\mathbb{R})$ into the larger space $\mathcal{M}(\mathbb{R})$ of finite measures, which is a Banach space for the total mass norm

$$\forall \mu \in \mathcal{M}(\mathbb{R}), \qquad \|\mu\|_{\mathcal{M}} := \int \mathrm{d}|\mu|(x).$$

In particular, $L^1(\mathbb{R})$ can be seen as the (closed) subspace of $\mathcal{M}(\mathbb{R})$ of finite measures which are absolutely continuous with respect to the Lebesgue measure. The space $\mathcal{M}(\mathbb{R})$ is the dual of the space $C_0(\mathbb{R})$ of continuous functions f that have limit $f(x) \longrightarrow 0$ at $x \to \pm\infty$. Note that the space $C_0(\mathbb{R})$ is separable, so that any bounded sequence of finite measures should have a weak accumulation point for the weak-$(*)$ convergence. In the case of the sequence (f_n) from Example 73, the convergence (A.3) shows that

$$f_n \overset{*}{\rightharpoonup} \delta_0 \qquad \text{in } \mathcal{M}(\mathbb{R}).$$

Example 78. In the case of Lebesgue spaces $L^p(\mathbb{R})$ for $1 < p < +\infty$, things are much more easier. Indeed, these spaces are reflexive, which means that they can be canonically[1] identified with their bi-dual $(L^p)'' = L^p$. For reflexive Banach spaces, the weak and weak-$(*)$ topologies are the same. Unfortunately, the spaces $\mathcal{M}(\mathbb{R})$ and $L^\infty(\mathbb{R})$ are not reflexive, which makes the notion of weak-$(*)$ topology indispensable.

Finally, as a conclusion to this discussion on weak convergence, we emphasize that the separability assumption for the predual in the Banach-Alaoglu Theorem 75 is absolutely necessary. Otherwise the the weak-$(*)$ topology is locally compact, but not sequentially locally compact. In order to stress this point, we give another series of examples.

Example 79. Consider the space $Y = \ell^\infty(\mathbb{N})$ of bounded sequences, which is not separable, and note $X = \ell^\infty(\mathbb{N})'$ its dual[2]. We define a sequence of bounded linear maps $T_n : \ell^\infty(\mathbb{N}) \longrightarrow \mathbb{R}$ by setting

$$\forall u \in \ell^\infty(\mathbb{N}), \qquad T_n(u) := u(n).$$

Then, the sequence (T_n) converges to $T \in \ell^\infty(\mathbb{N})'$ weakly-$(*)$ if and only if the limit

$$T_n(u) = u(n) \longrightarrow T(u)$$

[1] The adjective canonically means that the canonical embedding $L^p \longrightarrow (L^p)''$ is onto. There are non-reflexive Banach spaces X that are isomorphic to their bi-dual X'', but such that the canonical embedding is not onto.

[2] The dual space $\ell^\infty(\mathbb{N})$ is a very complicated and-non explicit Banach space. It contains the space $\ell^1(\mathbb{N})$ of summable sequences, by the canonical embedding, but the construction of elements $T \in \ell^\infty(\mathbb{N})'$ which are not in $\ell^1(\mathbb{N})$ requires the axiom of choice. It can be shown that $\ell^\infty(\mathbb{N})'$ is the space of finite measures on the Stone-Cech compactification $\beta\mathbb{N}$ of the natural numbers.

holds for all $u \in \ell^\infty(\mathbb{N})$. In other words, the convergence happens if and only if any bounded sequence $(u(n))$ has a limit at $n \to +\infty$, which is absurd. This shows that the sequence (T_n) does not have an accumulation point for the weak-$(*)$ convergence, although the weak-$(*)$ topology $\sigma\big(\ell^\infty(\mathbb{N})', \ell^\infty(\mathbb{N})\big)$ is (topologically) locally compact.

Example 80. We consider the sequence (T_n) from the previous example, but from a different point of view. Let $c_\ell(\mathbb{N})$ be the space of sequences u that are convergent: for any $u \in c_\ell(\mathbb{N})$, the limit $\lim_n u(n)$ exists. The space $c_\ell(\mathbb{N})$, equipped with the uniform norm $\|\,.\,\|_\infty$ is a separable Banach space, and the linear maps $T_n : c_\ell(\mathbb{N}) \longrightarrow \mathbb{R}$ are bounded. The sequence (T_n) converges weakly-$(*)$, and the limit T is given by

$$\forall u \in c_\ell(\mathbb{N}), \qquad T(u) := \lim_n u(n).$$

Example 81. Finally, we look at the space $\ell^1(\mathbb{N})$ of summable sequences. It is the dual of the space $c_0(\mathbb{N})$ of sequences u which converge to zero $u(n) \longrightarrow 0$ (equipped with the uniform norm). Since the space $c_0(\mathbb{N})$ is separable, $\ell^1(\mathbb{N})$ is locally sequentially compact for the weak-$(*)$ convergence. This is in sharp contrast with the space $L^1(\mathbb{R})$, although both spaces are Lebesgue spaces of integrable functions: the first one for functions $\mathbb{N} \longrightarrow \mathbb{R}$ with the counting measure, and the second one for functions $\mathbb{R} \longrightarrow \mathbb{R}$ with the Lebesgue measure.

A.4 Summary

Let us summarize the different ideas and types of convergence we have discussed so far.

- **In a Hilbert space:** any bounded sequence has a weak accumulation point. The weak topology is very nice. If only all spaces were Hilbert spaces.
- **The weak convergence** has nice properties, but the weak topology is usually not locally compact. This creates issues in spaces like $L^1(\mathbb{R})$.
- **Weak-$(*)$ convergence:** the topology is locally compact, and even locally sequentially compact if the predual is separable. The typical spaces in which weak-$(*)$ convergence is used are $L^\infty(\mathbb{R})$ and $\mathcal{M}(\mathbb{R})$. Bad things happen when the predual is not separable.
- **For reflexive spaces** the weak and weak-$(*)$ convergences are the same. For example, in $L^p(\mathbb{R})$ when $1 < p < +\infty$. Outside of Hilbert spaces, this is the most agreeable case.

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