# Seminar on Beyond Geometric Invariant Theory (after Daniel Halpern–Leistner)

Winter 2018/19, Tuesday 2-4, Room 0.001

The $D$-equivalence conjecture (after Bondol–Orlov [6]) claims that any two birationally equivalent projective Calabi–Yau manifolds are derived equivalent. Halpern–Leistner announced a proof of the D-equivalent conjecture for varieties which are birational to the moduli spaces of sheaves on a K3 surface. The proof is based on a program of Halpern–Leistner, called Beyond Geometric Invariant Theory [14], that generalize various concepts from Geometric Invariant Theory (GIT) to algebraic stacks. In this seminar starting from GIT quotients we learn about this program, with a focus on the proof of the announced result.

**Oct. 9: Overview and short motivation (omitted)**

**Oct. 23: GIT quotients and the Kempf–Ness stratification (Speaker: Lisa Li).**

Recall the construction of the quotient of a projective variety by a reductive group action in the sense GIT, see e.g. [16, Sec.2] and [7]. In particular the definition of the stable/semi-stable/unstable locus. Hilbert–Mumford criterion [16, Sec.2.1], [7, Sec.7]. Kempf–Ness stratification of a GIT quotient [9, Sec.2.1]. The goal is to state Theorem 1.1 of [9]. Another reference is Dolgachev-Hu.

**Oct. 30: The derived category of a GIT quotient (Speaker: Emma Brakkee).**

Discuss Theorem 1.1 of [9] (categorial Kirwan surjectivity) and its more refined version, Theorem 2.10. Explain the main steps in the proof following [9, Sec.3]. An alternative reference is [4].

**Nov. 6: Derived equivalences and variation of GIT (Speaker: Denis Nesterov).**

Following [9, Section 4] explain how Theorem 1.1 in [9] can be used to construct derived auto-equivalences between birational varieties obtained by a variation of GIT. In particular, Proposition 4.2. Discuss examples, for example the classical threefold flop, and Example 4.12. If time permits, explain also the results presented in [13].

**Nov. 13: Good moduli spaces (Speaker: Pieter Belmans).**

Recall the notions of fine and coarse moduli space associated to an Artin stack and the Keel-Mori criterion for DM stacks. Following [1], introduce the notion of good moduli space and explain its main properties (Theorem 4.16).

**Nov. 20: Universal property and examples of good moduli spaces (Speaker: Matthew Dawes).**

Following [1], discuss the universal property of good moduli spaces for maps to algebraic spaces, i.e. Theorem 6.6. Treat some examples, in particular Examples 8.3, 8.4, 8.7 and 8.9 in [1].

**Nov. 27: $\Theta$-stratifications (Speaker: Hacen Zelaci).**

We follow Section 1 of [10]. The $\Theta$-stratification is a generalization of the Harder–Narasimhan filtration to arbitrary stacks or moduli problems. Based on [10, Sec.1.1] start with the definition of Gieseker semistability and coherent sheaves, and explain how it naturally generalizes to the concept of $\Theta$-stratification. In particular, explain the correspondence between filtrations of coherent sheaves and maps from $\Theta = \mathbb{A}^1/\mathbb{G}_m$ into the stack of coherent sheaves. For this the easiest case is the stack of finite-dimensional vector spaces (equivalently, the stack of coherent sheaves on a point); the maps from $\Theta$ correspond here to filtered vector spaces. Definition of a numerical invariant [10, Def.1.12] and a Harder–Narasimhan filtration [10, Defn.1.15] of a point on the stack. The "Harder–Narasimhan problem". Section 1 of [10] is based on [11], see there for more details.

An additional reference for the talk is Section 1 of [15].

**Dec 4: $\Theta$-reductive stacks (Speaker: Hsueh-Yung Lin).**

Main source is Section 2 of [10]. Give the definition of $\Theta$-reductiveness (Definition 2.3) and discuss several examples, e.g. Examples 2.4, 2.5, 2.6. Present Proposition 2.9 and give an idea of the proof (Section 2.3). In particular define what it means for a numerical invariant $\mu$ to be bounded. Introduce $\Theta$-stratifications (Definition 2.21) and Theorem 2.24. In the example of the GIT quotient, explain how the $\Theta$-stratification reduces to the Kempf–Ness stratification. A more detailed reference is [11].

**Dec. 11: Luna étale slice theorem (Speaker: Gebhard Martin).**

The goal of the talk is to present the Luna étale slice theorem for algebraic stacks, which roughly says that every algebraic stack looks locally like a quotient stack. More precisely, every algebraic stack, locally of finite type over an algebraically closed field with affine stabilizers, is étale-locally a quotient stack in a neighborhood of a point with a linearly reductive stabilizer group [2]. This is a deep structure result and will be used in the next two talks at crucial points.

First recall the Luna étale sclice theorem for GIT quotients (of an affine variety by a linear reductive group) following the notes [8]. Then discuss and explain the statements of Theorems 1.1 and 1.2 of [2]. Give some examples and counterexamples (see [2]), and sketch the idea of the proof.

**Dec. 18: Existence of good moduli spaces (Speaker: Roberto Fringuelli)**

The goal of this talk is to discuss and explain the proof of the criterion of [3] for the existence of good moduli spaces. Details to follow.

**Jan. 8: Break.**

**Jan. 15: Semistable reduction for algebraic stacks (Speaker: Georg Oberdieck).** The paper [3] contains two main theorems: Semi-stable reduction for algebraic stacks, and the existence of good moduli spaces. The goal of this talk is to discuss and explain the former theorem. Since [3] is only a draft at this point, the precise details of these talks will appear later. [Several key points: Recall properties of the stack $\Theta = \mathbb{A}^1/\mathbb{G}_m$, introduce and discuss properties of the seperatedness test stack $ST_R$.]

**Jan. 22: Minimal model program for moduli of sheaves on K3 (Speaker: Thorsten Beckmann).**

Explain the background we need from [5] for the proof of the $D$-equivalence conjecture. More details to follow.

**Jan. 29:** Halpern-Leistner

# References


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